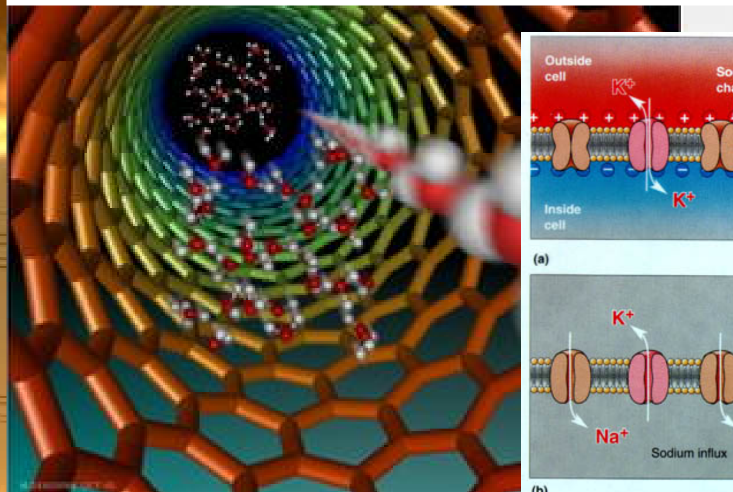
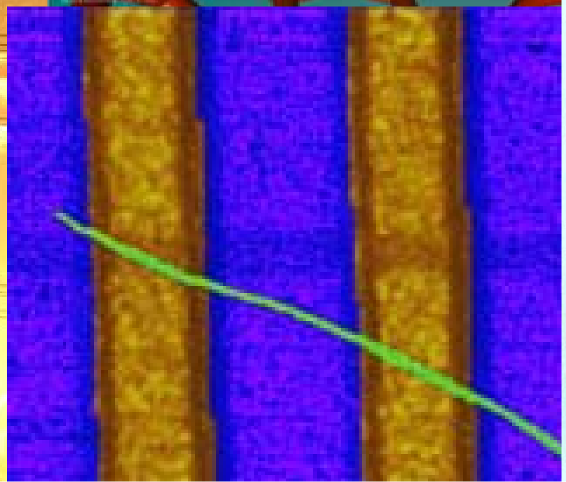
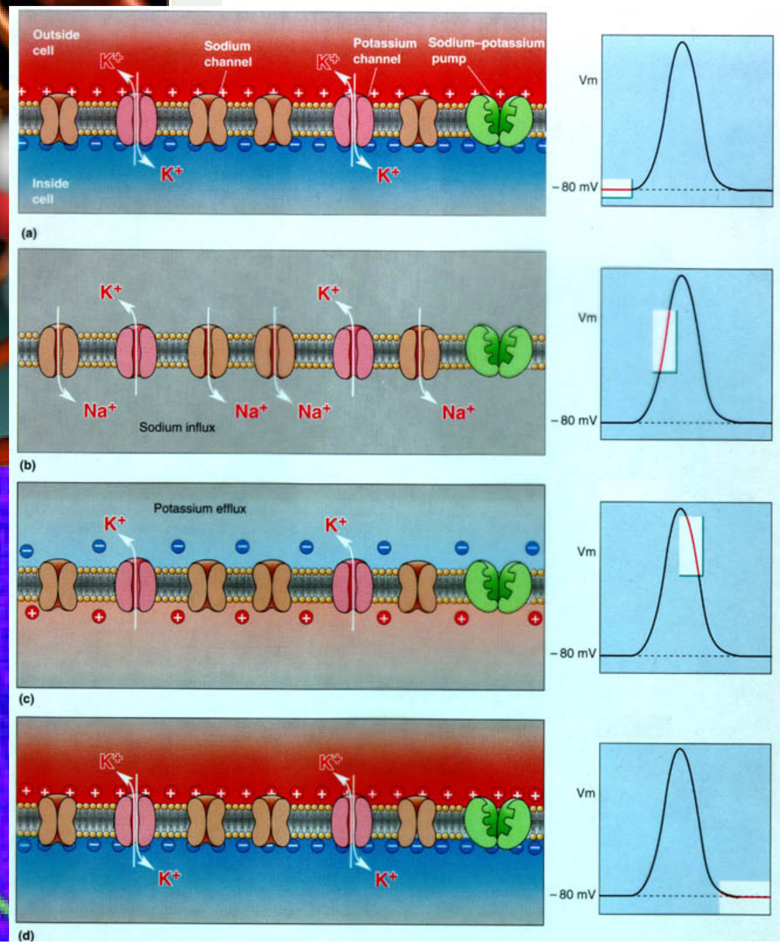
Chapter 1: Transmission Lines

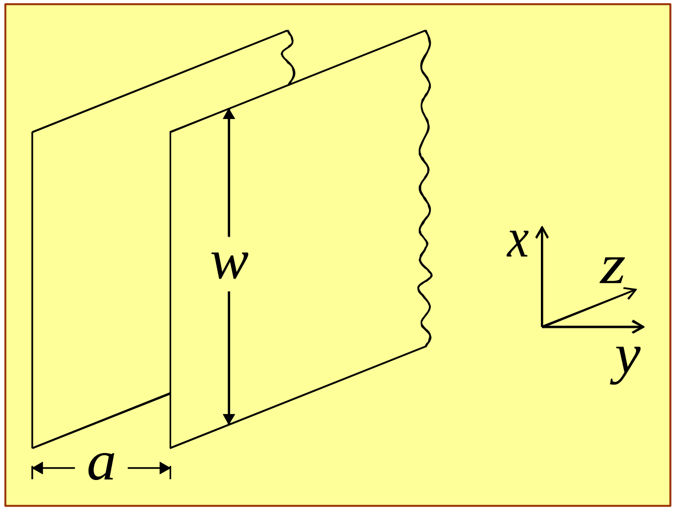
In this chapter we shall first recapitulate some of the topics learned in the framework of the course "Waves and Distributed Systems" and then we shall extend the analysis to topics that are of importance to microwave devices. But first a few examples:







1.1 Simple Model

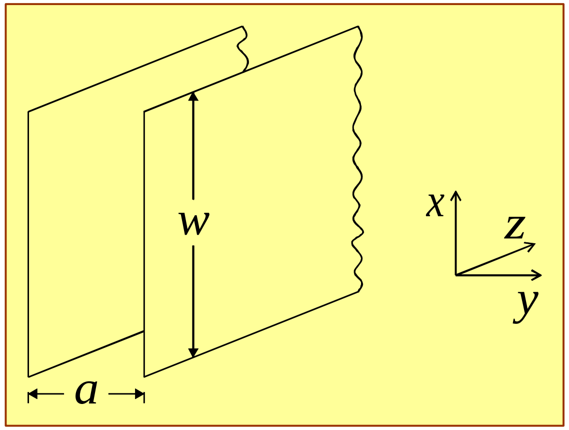


First we shall examine the propagation of an electromagnetic wave between two parallel plates located at a distance a one of the other as illustrated in the figure. The principal assumptions of this simple model are as follows:

1. No variation in the x direction i.e. $\partial_x = 0$.
2. Steady state e.g. $\exp(j\omega t)$.
3. The distance between the two plates (a) is very small so that even if there is any (field) variation in the y direction, it is negligible on the scale of the wavelength ($a \ll \lambda$)

$$\left|\frac{\partial}{\partial y}\right| \ll \left|\frac{\partial}{\partial z}\right| \Rightarrow \left|\frac{\partial}{\partial y}\right| \sim 0. \qquad (1.1.1)$$

4. The constitutive relations of the vacuum:
$\vec{D} = \varepsilon_0 \vec{E}$, $\quad \vec{B} = \mu_0 \vec{H}$ where $\mu_0 = 4\pi \times 10^{-7}$ [Henry/meter] and
$\varepsilon_0 = 8.85 \times 10^{-12}$ [Farad/Meter].



Based on the assumptions above, ME may be simplified.

(a) Gauss' law $\vec{\nabla}\cdot\vec{E}=0 \quad \rightsquigarrow \quad \partial_z E_z = 0 \Rightarrow E_z = const.$ we conclude that E_z is uniform between the two plates. Imposing next the boundary conditions on the two plates

$$E_z(y=0)=0 \qquad E_z(y=a)=0 \tag{1.1.2}$$

which means that the longitudinal electric field vanishes $(E_z \equiv 0)$.

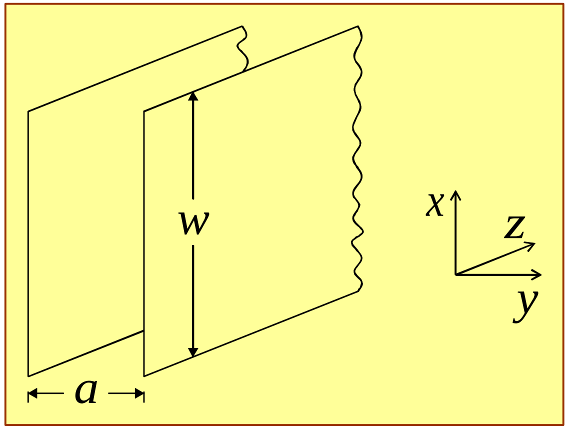
(b) In a similar way the magnetic induction satisfies $\vec{\nabla}\cdot\vec{B}=0$ and it may be shown that the longitudinal component of the magnetic induction vanishes $(B_z = 0)$.

(c) Faraday's equation reads $\nabla\times\vec{E} = -j\omega\vec{B}$ thus explicitly

$$\begin{vmatrix} 1_x & 1_y & 1_z \\ 0 & 0 & \partial_z \\ E_x & E_y & 0 \end{vmatrix} = -j\omega\vec{B} \Rightarrow \begin{array}{lcccc} 1_x & : & -\partial_z E_y & = & -j\omega B_x \\ 1_y & : & \partial_z E_x & = & -j\omega B_y \\ 1_z & : & 0 & = & 0 \end{array} \tag{1.1.3}$$

There is no variation in the y direction therefore since $E_x = 0$ for both $y=0$ and $y=a$, as in the case of E_z, we have $E_x \equiv 0$ therefore $B_y = 0$ thus

$$\boxed{\frac{\partial}{\partial z}E_y = j\omega\mu_o H_x.} \tag{1.1.4}$$



(d) Ampere's law reads $\vec{\nabla}\times\vec{H} = j\omega\vec{D}$, or explicitly taking advantage of the vanishing components we get

$$\begin{vmatrix} 1_x & 1_y & 1_z \\ 0 & 0 & \partial_z \\ H_x & 0 & 0 \end{vmatrix} = j\omega\vec{D} \Rightarrow \begin{array}{lcccc} \vec{1}_x & : & 0 & = & 0 \\ \vec{1}_y & : & \partial_z H_x & = & j\omega D_y \\ \vec{1}_z & : & 0 & = & 0 \end{array} \quad (1.1.5)$$

hence

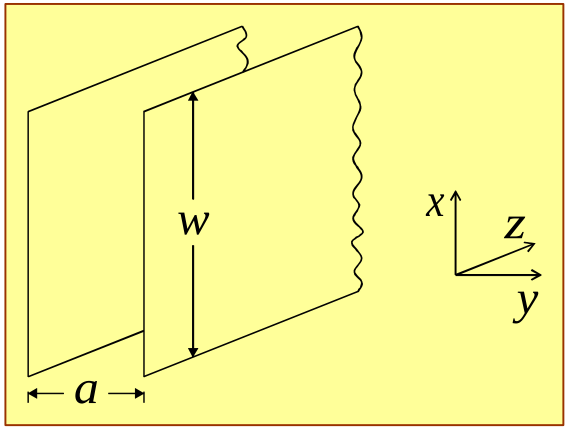
$$\boxed{\frac{\partial}{\partial z}H_x = j\omega\varepsilon_o E_y.} \qquad (1.1.6)$$

From these two equations [(1.1.4) and (1.1.6)] it can be readily seen that we obtain the **wave-equation** for each one of the components:

$$\left.\begin{array}{l} \dfrac{\partial}{\partial z}H_x = j\omega\varepsilon_o E_y \\ \dfrac{\partial}{\partial z}E_y = j\omega\mu_o H_x \end{array}\right\} \Rightarrow \boxed{\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right]E_y = 0} \qquad (1.1.7)$$

which has a solution of the form

$$E_y = A\exp\left(-j\frac{\omega}{c}z\right) + B\exp\left(j\frac{\omega}{c}z\right) \qquad (1.1.8)$$



It is convenient at this point to introduce the notation in terms of voltage and current. The voltage can be defined since $\oint \vec{E} \cdot d\vec{\ell} = 0$; it reads

$$V(z) = -E_y(z)a. \qquad (1.1.9)$$

In order to define the current we recall that based on the boundary conditions we have $\vec{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}$ where $\vec{K}$ is the surface current. Consequently, denoting by w the height of the metallic plates, the local current is $I = K_z w$ or

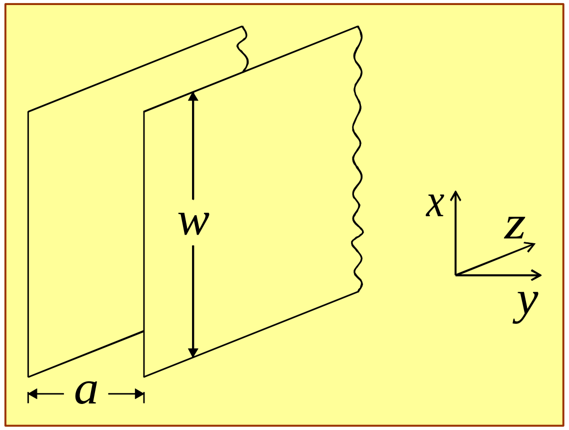
$$I(z) = H_x(z)w. \qquad (1.1.10)$$

Based on these two equations [(1.1.9)–(1.1.10)] it is possible to write

$$\begin{aligned}
&\frac{\partial}{\partial z}E_y = j\omega\mu_o H_x \Rightarrow \frac{\partial}{\partial z}\left[-\frac{V}{a}\right] = j\omega\mu_o \frac{I}{w} \Rightarrow \frac{\partial}{\partial z}V = -j\omega\left[\mu_o \frac{a}{w}\right]I \\
&\frac{\partial}{\partial z}H_x = j\omega\varepsilon_o E_y \Rightarrow \frac{\partial}{\partial z}\left[\frac{I}{w}\right] = j\omega\varepsilon_o\left[-\frac{V}{a}\right] \Rightarrow \frac{\partial}{\partial z}I = -j\omega\left[\varepsilon_o \frac{w}{a}\right]V
\end{aligned} \qquad (1.1.11)$$

The right hand side in both lines of (1.1.11) represent the so-called **transmission line equation** also known as **telegraph equations**.

$$\boxed{\begin{aligned}\frac{d}{dz}V(z) &= -j\omega L\, I(z)\\ \frac{d}{dz}I(z) &= -j\omega C\, V(z)\end{aligned}} \qquad (1.1.12)$$



C being the **capacitance per unit length** whereas L is the **inductance per unit length**; $L = \mu_o \dfrac{a}{w}$ and $C = \varepsilon_o \dfrac{w}{a}$. As expected, these two equations lead also to the wave equation

$$\left.\begin{aligned}\frac{dV}{dz} &= -j\omega L I(z)\\ \frac{dI}{dz} &= -j\omega C V(z)\end{aligned}\right\} \quad \left[\frac{d^2}{dz^2} + \beta^2\right] V(z) = 0 \qquad \boxed{\beta = \omega\sqrt{LC}} \qquad (1.1.13)$$

The general solution is $V(z) = Ae^{-j\beta z} + Be^{j\beta z}$ and correspondingly, the expression for the current is given by

$$I(z) = \frac{-1}{j\omega L}\frac{dV}{dz} = \frac{\beta}{\omega L}\left[Ae^{-j\beta z} - Be^{j\beta z}\right] \qquad (1.1.14)$$

defining the characteristic impedance $Z_c^{-1} = \dfrac{\beta}{\omega L} = \dfrac{\omega\sqrt{LC}}{\omega L}$ or $\boxed{Z_c \equiv \sqrt{\dfrac{L}{C}},}$ we get

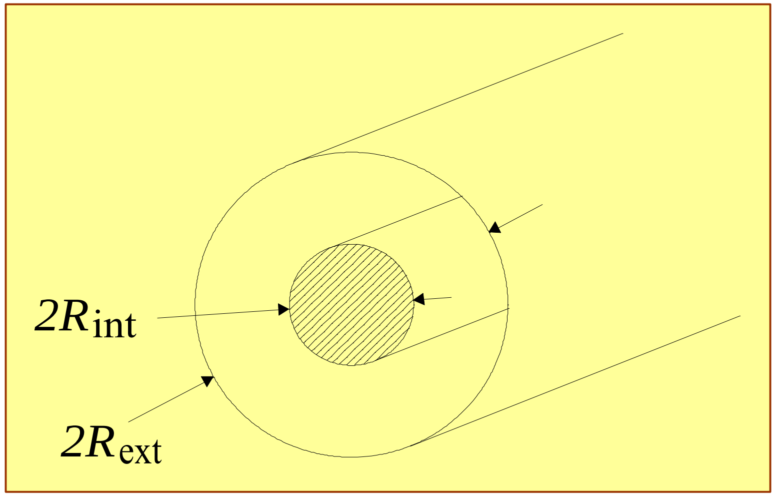
$$I(z)=\frac{1}{Z_c}\left[Ae^{-j\beta z}-Be^{+j\beta z}\right]. \qquad (1.1.15)$$

In the ***specific*** case under consideration

$$Z_c=\sqrt{\frac{\mu_o \frac{a}{w}}{\varepsilon_o \frac{w}{a}}}=\eta_o\frac{a}{w}\ ,\ \beta=\omega\sqrt{LC}=\frac{\omega}{c}. \qquad (1.1.16)$$

1.2 Coaxial Transmission Line

As indicated in the previous case, two parameters are to be determined: the capacitance per unit length (C) and the inductance per unit length (L). According to (1.1.11) these two parameters can be determined in static conditions. We determine next the **capacitance per unit length** of a coaxial structure. For this purpose it is assumed that on the inner wire a voltage V_o is applied, whereas the outer cylinder is grounded.



Consequently, the potential is given by

$$\varphi(r) = V_o \frac{\ln(r / R_{ext})}{\ln(R_{int} / R_{ext})}. \tag{1.2.1}$$

and the corresponding electric field associated with this potential is

$$E_r = -\frac{\partial \varphi}{\partial r} = -V_o \frac{1}{r} \frac{1}{\ln(R_{int} / R_{ext})}. \tag{1.2.2}$$

The charge per unit surface at $r = R_{ext}$ is calculated based on $\vec{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$ and it is given by

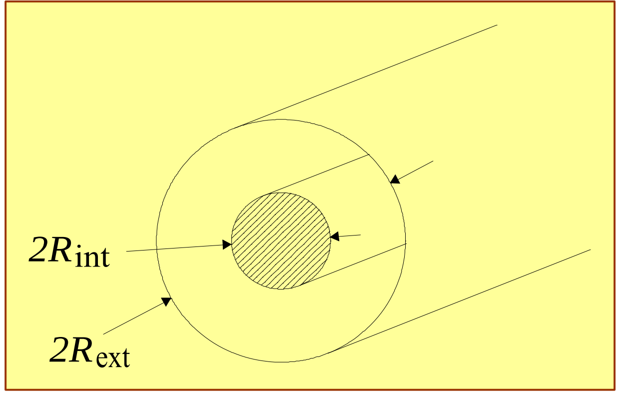
$$\rho_s = \varepsilon_0 \frac{V_0}{R_{ext}} \frac{1}{\ln\left(R_{ext} / R_{int}\right)}. \tag{1.2.3}$$

Based on this result, the charge per unit length (Δ_z) may be expressed as

$$\frac{Q}{\Delta_z} = \rho_s 2\pi R_{ext} = \varepsilon_0 \frac{V_o}{R_{ext}} \frac{1}{\ln\left(\frac{R_{ext}}{R_{int}}\right)} 2\pi R_{ext} = \frac{\varepsilon_o V_o 2\pi}{\ln\left(R_{ext}/R_{int}\right)}. \tag{1.2.4}$$

Consequently, the capacitance per unit length is given by

$$\boxed{C \equiv \frac{Q/\Delta_z}{V_o} = \frac{2\pi\varepsilon_o}{\ln(R_{ext}/R_{int})}.} \qquad (1.2.5)$$



In a similar way, we shall calculate the **inductance per unit length**. Assuming that the inner wire carries a current I, based on Ampere law the azimuthal magnetic field is

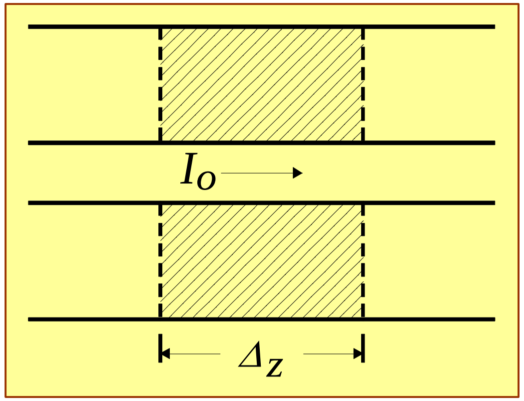
$$H_\phi(r) = \frac{I_o}{2\pi r}. \qquad (1.2.6)$$

With this expression for the magnetic field, we can calculate the magnetic flux. It is given by

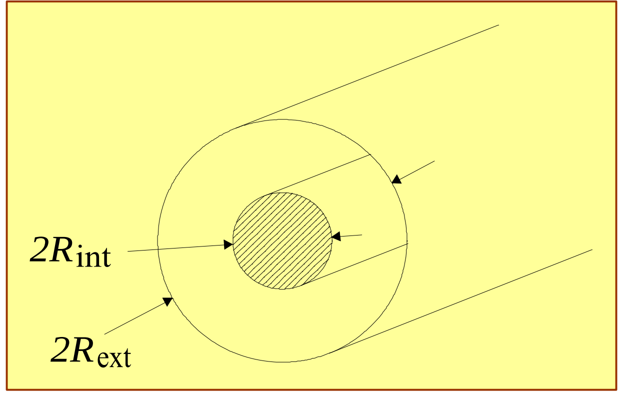
$$\Phi = \mu_o \Delta_z \int_{R_{int}}^{R_{ext}} dr H_\phi(r) = \mu_o \Delta_z \frac{I}{2\pi} \ln \frac{R_{ext}}{R_{int}}. \qquad (1.2.7)$$

The inductance per unit length Δ_z is

$$\boxed{L \equiv \frac{\Phi/\Delta_z}{I} = \frac{\mu_o}{2\pi} \ln\left(\frac{R_{ext}}{R_{int}}\right).} \qquad (1.2.8)$$

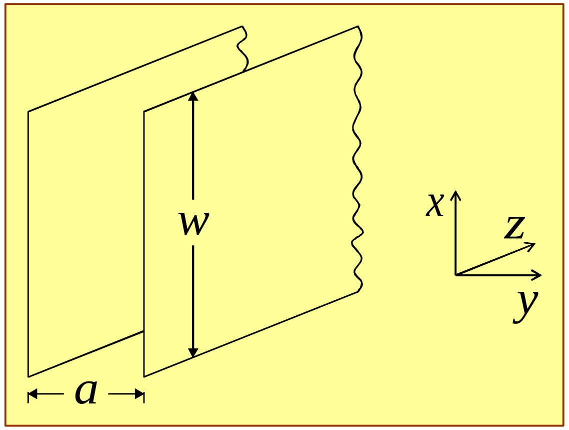


To summarize the parameters of a coaxial transmission line



$$\boxed{\begin{aligned} Z_c &= \sqrt{\frac{L}{C}} = \eta_o \frac{1}{2\pi} \ln\left(\frac{R_{ext}}{R_{int}}\right) \\ \beta &= \frac{\omega}{c} \end{aligned}} \qquad (1.2.9)$$

Exercise 1.1: Determine Z_c and β for a coaxial line filled with a material (ε_r, μ_r)?



1.3 Low Loss System

Based on Ampere's law we obtained

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon_o\varepsilon\vec{E} \rightsquigarrow \frac{d}{dz}I(z) = -j\omega CV(z),$$

(1.3.1)

where we assumed a line without dielectric (ε_r) and Ohm (σ) loss. In the case of dielectric loss we have

$$\varepsilon_r = \varepsilon' - j\varepsilon'' \quad (1.3.2)$$

or in our case

$$j\omega C \to j\omega C + G \equiv Y. \quad (1.3.3)$$

In a similar way based on Faraday's law

$$\vec{\nabla} \times \vec{E} = -j\omega\mu_o\mu_r\vec{H} \Rightarrow \frac{d}{dz}V(z) = -j\omega LI(z) \quad (1.3.4)$$

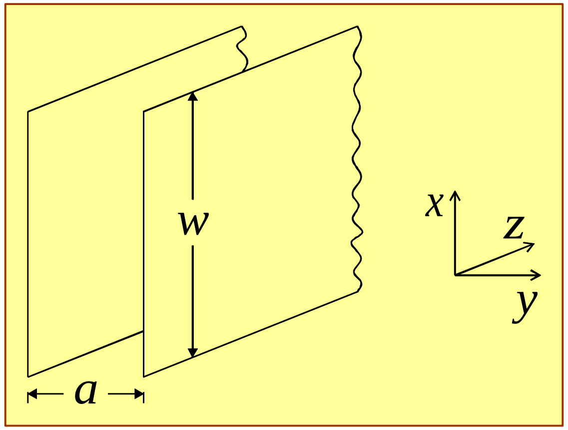
and the magnetic losses

$$\mu_r = \mu_r' - j\mu_r''$$

allows us to extend the definition according to

$$j\omega L \rightsquigarrow j\omega L + R \equiv Z \quad (1.3.5)$$

hence the equations



$$\frac{d}{dz}I(z) = -YV(z)$$
$$\frac{d}{dz}V(z) = -ZI(z) \tag{1.3.6}$$

may be conceived as a generalization of the transmission line equations in the presence of loss. The characteristic impedance for small loss line is

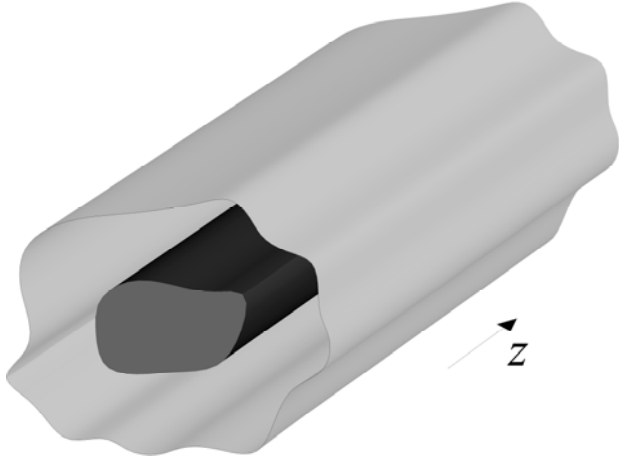
$$Z_c = \sqrt{\frac{Z}{Y}} \simeq \sqrt{\frac{L}{C}}\left[1 + j\left(\frac{G}{2\omega C} - \frac{R}{2\omega L}\right)\right] \tag{1.3.7}$$

and the wave number, assuming a solution of the form $\exp(-\gamma z)$,

$$\gamma = \alpha + j\beta$$
$$\alpha \simeq \frac{R}{2Z_o} + \frac{GZ_o}{2} \tag{1.3.8}$$
$$\beta \simeq \omega\sqrt{LC}\left[1 - \frac{RG}{4\omega^2 LC} + \frac{G^2}{8\omega^2 C^2} + \frac{R^2}{8\omega^2 L^2}\right]$$

Exercise 1.2: Prove the relations in Eq. (1.3.8).

1.4 Generalization of the Transmission Line Equations



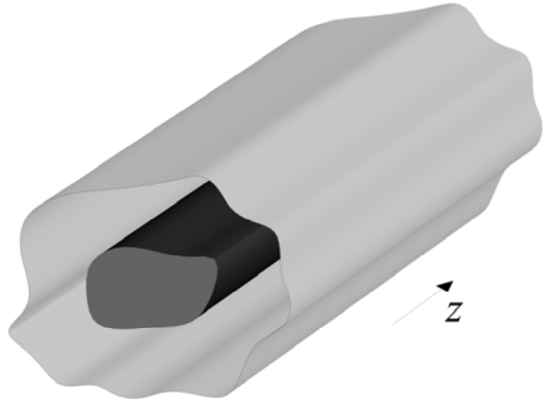
The fundamental assumptions of the analysis are:
(i) TEM, (ii) the wave propagates in the z direction, (iii) we distinguish between longitudinal (z) and transverse ($\perp$) components $\vec{\nabla} = \vec{\nabla}_{\perp} + \frac{\partial}{\partial z}\vec{1}_z$.

From Faraday law, $\vec{\nabla} \times \vec{E} = -j\omega\mu_o\mu_r\vec{H}$, we obtain

$$\begin{vmatrix} 1_x & 1_y & 1_z \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & 0 \end{vmatrix} = \begin{matrix} \vec{1}_x(-) & \partial_z E_y \\ \vec{1}_y & \partial_z E_x \\ \vec{1}_z & \partial_x E_y - \partial_y E_x \end{matrix} \tag{1.4.1}$$

thus

$$\left.\begin{matrix} 1_x: & -\partial_z E_y = -j\omega\mu_o\mu_r H_x \\ 1_y: & +\partial_z E_x = -j\omega\mu_o\mu_r H_y \\ 1_z: & \partial_x E_y - \partial_y E_x = 0 \end{matrix}\right\} \quad \begin{matrix} \vec{1}_{\perp}: & \vec{1}_z \times \dfrac{\partial \vec{E}_{\perp}}{\partial z} = -j\omega\mu_o\mu_r\vec{H}_{\perp} \\ \vec{1}_z: & \vec{\nabla}_{\perp} \times \vec{E}_{\perp} = 0. \end{matrix} \tag{1.4.2}$$



In a similar way, from Ampere's law we have

$$\vec{\nabla}\times\vec{H} = +j\omega\varepsilon_o\varepsilon_r\vec{E} \Rightarrow \begin{cases} \vec{1}_\perp : \vec{1}_z \times \dfrac{\partial \vec{H}_\perp}{\partial z} & = \quad j\omega\varepsilon_o\varepsilon_r\vec{E}_\perp \\ \vec{1}_z : \vec{\nabla}_\perp \times \vec{H}_\perp & = \quad 0. \end{cases} \tag{1.4.3}$$

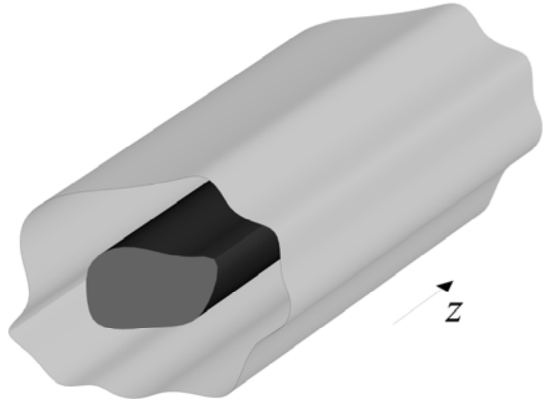
From the two curl equations

$$\begin{aligned} \vec{\nabla}_\perp \times \vec{E}_\perp &= 0 \quad \Rightarrow \quad \vec{E}_\perp = g(z)\nabla_\perp\varphi(x,y) \quad \Rightarrow \quad \nabla_\perp^2\varphi = 0 \\ \vec{\nabla}_\perp \times \vec{H}_\perp &= 0 \quad \Rightarrow \quad \vec{H}_\perp = h(z)\nabla_\perp\psi(x,y) \quad \Rightarrow \quad \nabla_\perp^2\psi = 0, \end{aligned} \tag{1.4.4}$$

we conclude that the *transverse variations* of the transverse field components are determined by **2D Laplace equation** justifying the use of DC quantities adopted above (capacitance and inductance per unit length). From the other two equations we get the wave equation

$$\begin{aligned} &\frac{\partial}{\partial z}\left[\vec{1}_z \times \frac{\partial \vec{E}_\perp}{\partial z}\right] = -j\omega\mu_o\mu_r\frac{\partial \vec{H}_\perp}{\partial z} \\ &\vec{1}_z \times \left[\vec{1}_z \times \frac{\partial^2 \vec{E}_\perp}{\partial z^2}\right] = -j\omega\mu_o\mu_r\left[\vec{1}_z \times \frac{\partial \vec{H}_\perp}{\partial z}\right] = -j\omega\mu_o\mu_r[j\omega\varepsilon_o\varepsilon_r\vec{E}_\perp], \end{aligned} \tag{1.4.5}$$

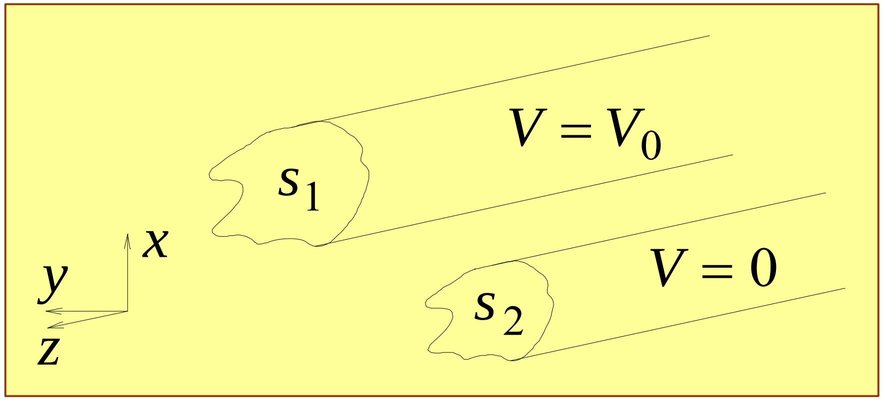
or explicitly



$$\boxed{\left[\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c^2}\mu_r\varepsilon_r\right]\vec{E}_\perp=0.} \tag{1.4.6}$$

The last equation determines the dynamics of $g(z)$ [see (1.4.4)] and the solution has the form $g(z)=e^{-j\beta z}$ where $\beta=(\omega/c)\sqrt{\varepsilon_r\mu_r}$. Note that

$$\begin{aligned}
&\vec{1}_z\times\frac{\partial\vec{E}_\perp}{\partial z}=-j\omega\mu_o\mu_r\vec{H}_\perp \Rightarrow \vec{1}_z\times\vec{E}_\perp=\sqrt{\frac{\mu_o}{\varepsilon_o}\frac{\mu_r}{\varepsilon_r}}\vec{H}_\perp \\
&\vec{1}_z\times\frac{\partial\vec{H}_\perp}{\partial z}=j\omega\varepsilon_o\varepsilon_r\vec{E}_\perp \Rightarrow \vec{1}_z\times\vec{H}_\perp=-\sqrt{\frac{\varepsilon_o}{\mu_o}\frac{\varepsilon_r}{\mu_r}}\vec{E}_\perp
\end{aligned} \tag{1.4.7}$$



As in the previous two cases we shall see next how the electric parameters can be calculated in the general case and for the sake of simplicity we shall assume that the medium has uniform transverse and longitudinal properties. The electric field in the entire space is given by $\vec{E}_{\perp} = -(\nabla_{\perp}\varphi)e^{-j\beta z}$ whereas the magnetic field is

$$\vec{H}_{\perp} = \sqrt{\frac{\varepsilon_o}{\mu_o}\frac{\varepsilon_r}{\mu_r}}(-\vec{1}_z \times \nabla_{\perp}\varphi)e^{-j\beta z}. \quad (1.4.8)$$

Note that associated to this electric field, one can define the voltage

$$V_o = -\int_{s_1}^{s_2} \vec{E}_{\perp} \cdot d\vec{\ell} = \int_{s_1}^{s_2} \nabla_{\perp}\varphi \cdot d\vec{\ell} = \int_{s_1}^{s_2} d\varphi \quad (1.4.9)$$

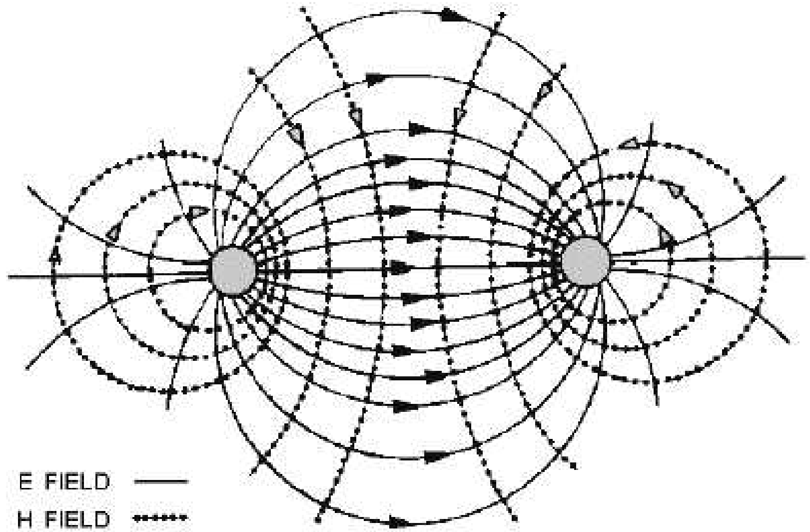
such that $V(z) = V_o e^{-j\beta z}$. On the two (ideal) conductors the electric field generates a surface charge given by

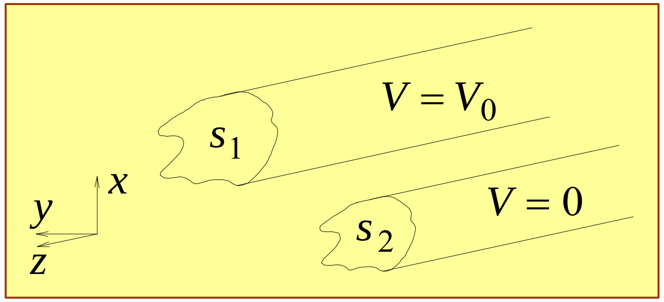
$$\rho_s = \varepsilon_o \varepsilon_r \vec{n} \cdot \vec{E}_{\perp}, \quad (1.4.10)$$

therefore the charge per unit length is

$$\frac{Q}{\Delta_z} = \oint dl\, \rho_s = \oint dl\, \varepsilon_o \varepsilon_r (\vec{n} \cdot \vec{E}_{\perp}). \quad (1.4.11)$$

Since by virtue of linearity of Maxwell's equations





the charge per unit length is proportional to the applied voltage $\frac{Q}{\Delta_z} = CV_0$ we get

$$C = \varepsilon_o \varepsilon_r \frac{1}{V_0} \oint dl (\vec{n} \cdot \vec{E}_\perp). \tag{1.4.12}$$

In a similar way, the magnetic field generates on the metallic electrode (wire) a surface current given by

$$\vec{J}_s = \vec{n} \times \vec{H}_\perp. \tag{1.4.13}$$

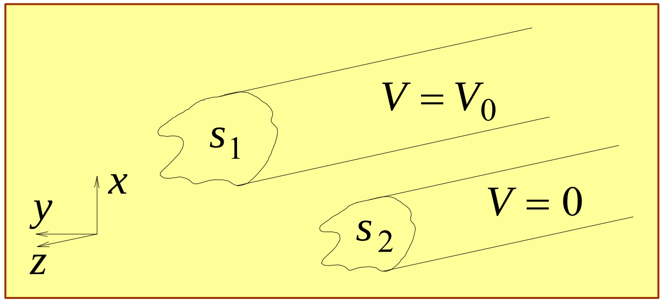
Since it was shown that $\vec{H}_\perp = \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o \mu_r}} \vec{1}_z \times \vec{E}_\perp$ we conclude that

$$\vec{J}_s = \vec{n} \times \vec{H}_\perp = \vec{n} \times [\vec{1}_z \times \vec{E}_\perp] \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o \mu_r}} = \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o \mu_r}} (\vec{n} \cdot \vec{E}_\perp) \vec{1}_z \tag{1.4.14}$$

hence the total current is

$$I_0 = \oint \vec{J}_s \cdot \vec{1}_z \, dl = \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o \mu_r}} \oint dl \left(\vec{n} \cdot \vec{E}_\perp \right) \tag{1.4.15}$$

At this point rather than calculating the inductance per unit length we combine the previous result for the charge per unit length and (1.4.15) the result being



$$\frac{I_o}{Q/\Delta_z} = \frac{\sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o \mu_r}} \oint dl\, \vec{n}\cdot\vec{E}_\perp}{\varepsilon_o \varepsilon_r \oint dl\, \vec{n}\cdot\vec{E}_\perp} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}. \qquad (1.4.16)$$

However, having established this relation between the current and the charge per unit length we may use again the linearity of Maxwell's equations and express $Q/\Delta_z = CV_0$. Substituting in Eq. (1.4.16) we get

$$\frac{I_0}{CV_o} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \qquad (1.4.17)$$

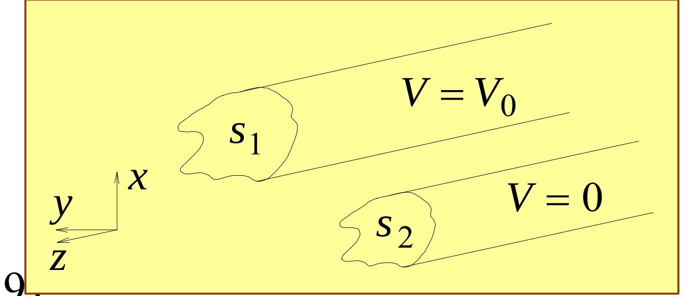
but by definition

$$\frac{V_0}{I_0} = Z_c, \qquad (1.4.18)$$

which finally implies that

$$\boxed{\frac{1}{CZ_c} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}.}$$

This result leads to a very important conclusion namely, **in a transmission line of *uniform* electromagnetic properties it is sufficient to calculate the capacitance per unit length.** Bearing in mind that $Z_c = \sqrt{L/C}$ we find that once C is established,



$$L = \frac{\mu_r \varepsilon_r}{Cc^2}. \qquad (1.4.19)$$

It is important to re-emphasize that this relation is valid only if the electromagnetic properties (ε_r, μ_r) are *uniform* over the cross-section.

Exercise 1.3: Calculate the capacitance per unit length of two wires of radius R which are at a distance $d > 2R$ apart.

Another quantity that warrants consideration is the **average power**

$$\begin{aligned} P &= \frac{1}{2}\mathrm{Re}\left\{\int dxdy\left(\vec{E}_\perp \times \vec{H}_\perp^*\right)\cdot \vec{1}_z\right\} = \frac{1}{2}\mathrm{Re}\left\{\int dxdy\left[\vec{E}_\perp \times \left(\vec{1}_z \times \vec{E}_\perp^*\right)\right]\cdot \vec{1}_z\right\}\sqrt{\frac{\varepsilon_o\varepsilon_r}{\mu_o\mu_r}} \\ &= \frac{1}{2}\mathrm{Re}\left\{\int dxdy\,\vec{E}_\perp \cdot \vec{E}_\perp^*\right\}\sqrt{\frac{\varepsilon_o\varepsilon_r}{\mu_o\mu_r}} = \frac{2}{\varepsilon_o\varepsilon_r}\sqrt{\frac{\varepsilon_o\varepsilon_r}{\mu_o\mu_r}}\left[\frac{1}{4}\int dxdy\,\varepsilon_o\varepsilon_r \vec{E}_\perp \cdot \vec{E}_\perp^*\right] \qquad (1.4.20) \\ &= \frac{2}{\sqrt{\varepsilon_o\mu_o\varepsilon_r\mu_r}}W_e = \frac{c}{\sqrt{\varepsilon_r\mu_r}}[W_e + W_m]. \end{aligned}$$

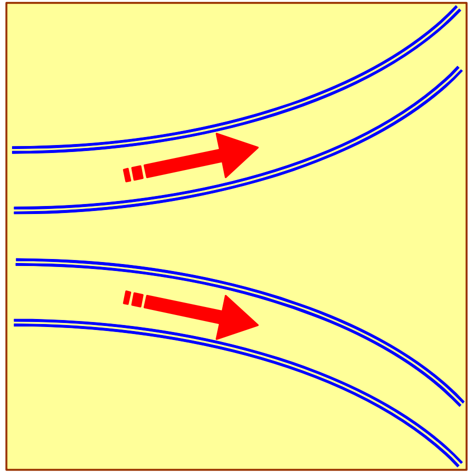
Exercise 1.4: In the last expression we used the fact that $W_e = W_m$ -- prove it.

Exercise 1.5: Show that the power can be expressed as $P = \frac{1}{2} Z_c \mid I_o \mid^2 = \frac{1}{2} V_o I_o^*$.

Finally, we may define the energy velocity as the average power propagating along the transmission line over the total average energy per unit length

$$V_{\text{en}} = \frac{P}{W_e + W_m} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}.$$

Exercise 1.6: Show that the material is not frequency dependent, this quantity equals exactly the group velocity. What if not? Namely $\varepsilon_{\text{r}}(\omega)$.



1.5 Non-Homogeneous Transmission Line

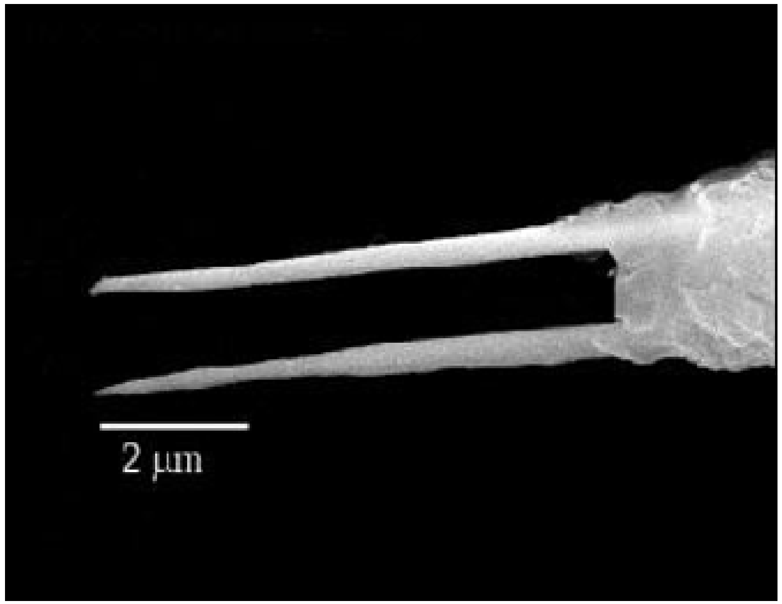
There are cases when either the electromagnetic properties or the geometry vary along the structure. In these cases the impedance per unit length (Z) and admittance per unit length (Y) are z-dependent i.e.

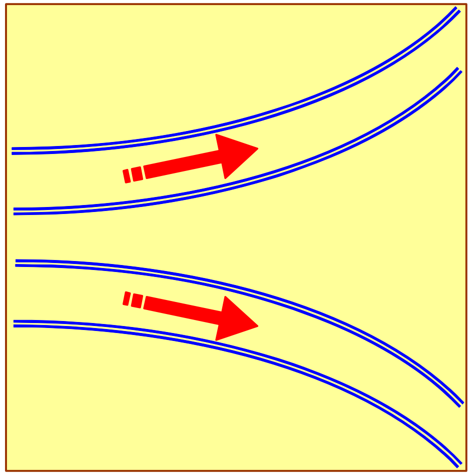
$$\frac{dV(z)}{dz} = -Z(z)I(z)$$
$$\frac{dI(z)}{dz} = -Y(z)V(z). \qquad (1.5.1)$$

As a result, the voltage or current satisfy an equation that to some extent differs from the regular wave equation

$$\frac{d^2V}{dz^2} = -\frac{dZ(z)}{dz}I(z) - Z(z)\frac{dI(z)}{dz}$$
$$= \left\{\frac{d}{dz}\ln\left[Z(z)\right]\right\}\frac{dV}{dz} + Z(z)Y(z)V(z)$$

A solution of a general character is possible only using numerical methods. However, an analytic solution is possible if we assume an "exponential" behavior of the





form

$$Z(z) = j\omega L_o \exp(qz)$$
$$Y(z) = j\omega C_o \exp(-qz) \qquad (1.5.3)$$

Substituting these expressions in Eq. (1.5.2) we get

$$\frac{d^2V(z)}{dz^2} - q\frac{dV}{dz} + \omega^2 L_o C_o V = 0 \qquad (1.5.4)$$

therefore assuming a solution of the form $V(z) = V_o e^{-\gamma_1 z}$ we conclude that

$$\gamma_1 = -\frac{1}{2}\left[q \pm \sqrt{q^2 - 4\omega^2 L_o C_o}\right] = -\frac{1}{2}\left[q - \sqrt{q^2 - 4\omega^2 L_o C_o}\right]. \qquad (1.5.5)$$

In a similar way, the equation for the current is given by

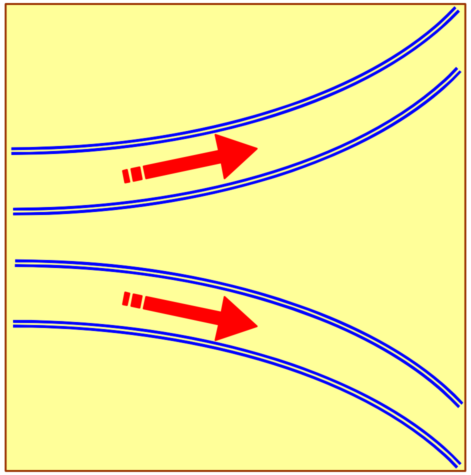
$$\frac{d^2I}{dz^2} - \frac{dI}{dz}\frac{d}{dz}\ln[Y(z)] - YZI(z) = 0. \qquad (1.5.6)$$

Assuming a solution of the form $I = I_o e^{-\gamma_2 z}$ we obtain

$$\gamma_2 = \frac{1}{2}\left[q \pm \sqrt{q^2 - 4\omega^2 L_o C_o}\right] = \frac{1}{2}\left[q + \sqrt{q^2 - 4\omega^2 L_o C_o}\right]. \qquad (1.5.7)$$

It is convenient to define

$$\omega_c^2 \equiv \frac{q^2}{4L_0C_0}, \qquad (1.5.8)$$



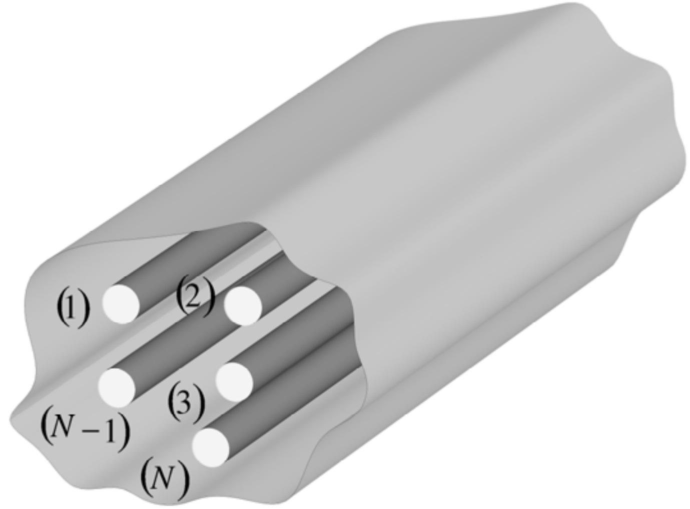
which sets a "cut-off" in the sense that for $\omega < \omega_c$ both γ_1 and γ_1 are real. The second important result is that the impedance along the transmission line

$$\frac{V(z)}{I(z)} = \frac{V_0 e^{-\gamma_1 z}}{I_0 e^{-\gamma_2 z}} = Z_c(0) e^{qz} \tag{1.5.9}$$

is *frequency independent*.

Exercise 1.7: Plot the average power along such a transmission line as well as the average electric and magnetic energies. What is the energy velocity?

1.6 Coupled Transmission Lines



Microwave or high frequency circuits consist typically of many elements connected usually with wires that may be conceived as transmission lines. The proximity of one line to another may lead to **coupling phenomena**. Our purpose in this section is to formulate the telegraph equations in the presence of coupling. With this purpose in mind let us assume N transmission lines each one of which is denoted by an index $n(=1,2\cdots N)$ -- as illustrated in the figure above. Ignoring loss in the system we may conclude that the relation between the charge per unit length of each "wire" is related to the voltages by

$$\frac{Q_{\nu}}{\Delta_z}=\sum_{n=1}^{N}C_{\nu n}V_n \qquad (1.6.1)$$

$C_{\nu,n}$ being the capacitance matrix per unit length. In a similar way, it is possible to establish the inductance matrix per unit length relating the voltage on wire ν with all the currents

$$\frac{\Phi_{\nu}}{\Delta_z}=\sum_{n=1}^{N}L_{\nu n}I_n. \qquad (1.6.2)$$

Having these two equations [(1.6.1)–(1.6.2)] in mind, we may naturally extend the telegraph equations to read

$$\frac{d}{dz}V_\nu(z) = -j\omega\sum_{n=1}^{N} L_{\nu n} I_n(z) \qquad\qquad \frac{d}{dz}I_\nu(z) = -j\omega\sum_{n=1}^{N} C_{\nu n} V_n(z). \tag{1.6.3}$$

Subsequently we discuss in more detail phenomena linked to this coupling process however, at this point we wish to emphasize that the number of wave-numbers (β^2) corresponds to the **number of ports**. This is evident since

$$\vec{V}(z) = \vec{V}_0 e^{-j\beta z} \qquad \vec{I}(z) = \vec{I}_0 e^{-j\beta z}, \tag{1.6.4}$$

enabling to simplify (1.6.3) to read

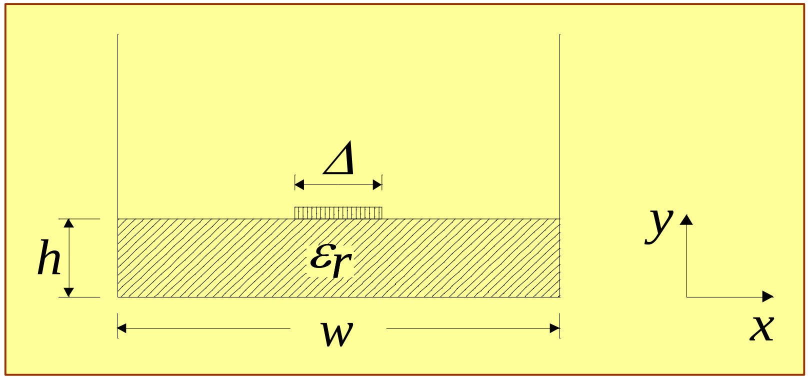
$$\beta\vec{V}_0 = \omega \underline{\underline{L}}\vec{I}_0 \qquad \beta\vec{I}_0 = \omega\underline{\underline{C}}\vec{V}_0 \tag{1.6.5}$$

thus the wavenumber is the non-trivial solution of

$$[\omega^2 \underline{\underline{L}}\,\underline{\underline{C}} - \beta^2 \underline{\underline{\delta}}]\vec{V}_0 = 0 \quad \text{or} \quad [\omega^2 \underline{\underline{C}}\,\underline{\underline{L}} - \beta^2 \underline{\underline{\delta}}]\vec{I}_0 = 0 \tag{1.6.6}$$

wherein $\underline{\underline{\delta}}$ is the unity matrix. Clearly the normalized wave number $\bar{\beta}^2 \equiv (\beta c/\omega)^2$ are the **eigen-values of the matrix** $\underline{\underline{L}}\,\underline{\underline{C}}$ ($= \underline{\underline{C}}\,\underline{\underline{L}}$ since both matrices are symmetric) and if the dimension of $\underline{\underline{C}}$ and $\underline{\underline{L}}$ is N then, the number of the eigen wavenumbers is also N.

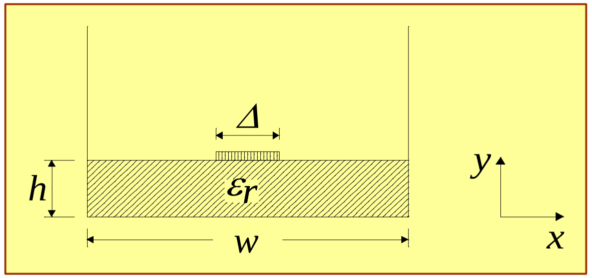
1.7 Microstrip



In this section we shall discuss in some detail some of the properties of a microstrip which is an essential component in any micro-electronic as well as microwave circuit. The microstrip consists of a thin and narrow metallic strip located on a thicker dielectric layer. On the other side of the latter, there is a ground metal; the side walls have been introduced in order to simplify the analysis and the width w is large enough such that it does not affect the physical processes in the vicinity of the strip. We shall examine a simplified model of this system and for this purpose we make the following **assumptions**: (i) The width of the device is much larger than the height ($w \gg h$) and the width ($w \gg \Delta$) of the strip.

(ii) The charge on the strip is distributed uniformly.

Our goal is to calculate the two parameters of the transmission line: **capacitance and inductance per unit length**. With this purpose in mind we shall start with evaluation of the capacitance therefore let us assume a general charge distribution on the



strip

$$\rho_s(x) = \begin{cases} \eta(x) & |x - \frac{w}{2}| < \frac{\Delta}{2} \\ 0 & |x - \frac{w}{2}| > \frac{\Delta}{2} \end{cases} \qquad (1.7.1)$$

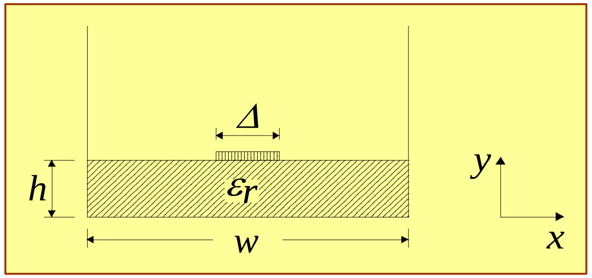
With the exception of $y = h$ the potential is given by

$$\phi(x,y) = \begin{cases} \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{w}\right) \sinh\left(\frac{\pi n y}{w}\right) & 0 \le y \le h \\ \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n x}{w}\right) e^{-\frac{\pi n}{w}(y-h)} & y \ge h. \end{cases} \qquad (1.7.2)$$

The continuity of the potential at $y = h$ implies

$$\phi(x, y = h) = \sum_n A_n \sin\left(\pi n \frac{x}{w}\right) \sinh\left(\frac{\pi n h}{w}\right) = \sum_n B_n \sin\left(\frac{\pi n x}{w}\right)$$

$$\Rightarrow \quad \boxed{A_n \sinh\left(\frac{\pi n h}{w}\right) = B_n}. \qquad (1.7.3)$$



The electric induction D_y is discontinuous at this plane. In each one of the two regions the field is given by

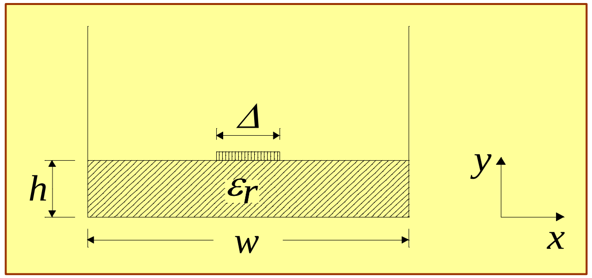
$$
\begin{aligned}
D_y(x, y<h) &= -\varepsilon_o \varepsilon_r \sum_n A_n \left(\frac{\pi n}{w}\right) \sin\left(\frac{\pi n x}{w}\right) \cosh\left(\frac{\pi n y}{w}\right) \\
D_y(x, y>h) &= \varepsilon_o \sum_n B_n \left(\frac{\pi n}{w}\right) \sin\left(\frac{\pi n x}{w}\right) \exp\left[-\frac{\pi n}{w}(y-h)\right].
\end{aligned}
\qquad (1.7.4)
$$

With these expressions, we can write the boundary conditions i.e., $\vec{n}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s$ in the following form

$$
\varepsilon_o \sum_n \left(\frac{\pi n}{w}\right) \sin\left(\frac{\pi n x}{w}\right) \left[B_n + \varepsilon_r A_n \cosh\left(\frac{\pi n h}{w}\right)\right] = \begin{cases} \eta(x) & |x - \frac{w}{2}| < \frac{\Delta}{2} \\ 0 & |x - \frac{w}{2}| > \frac{\Delta}{2}. \end{cases} \qquad (1.7.5)
$$

Using the orthogonality of the $\sin$ function we obtain [for this reason the two side walls were introduced]

$$
\boxed{B_n + \varepsilon_r A_n \cosh\left(\frac{\pi n h}{w}\right) = \frac{1}{\varepsilon_o} \frac{2}{\pi n} \int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dx \eta(x) \sin\left(\frac{\pi n x}{w}\right).} \qquad (1.7.6)
$$



The next step is to substitute (1.7.3) into the last expression. The result is

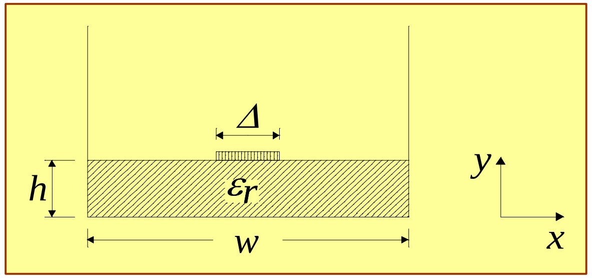
$$A_n\left[\sinh\left(\frac{\pi nh}{w}\right)+\varepsilon_r\cosh\left(\frac{\pi nh}{w}\right)\right]=\frac{1}{\varepsilon_o}\frac{2}{\pi n}\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}}dx\eta(x)\sin\left(\frac{\pi nx}{w}\right). \qquad (1.7.7)$$

Consequently, subject to the assumption that $\eta(x)$ is known, the potential is known in the entire space and specifically at $y=h$ is given by

$$\phi(x,y=h)=\sum_n \sin\left(\frac{\pi nx}{w}\right)\frac{1}{1+\varepsilon_r \mathrm{ctanh}\left(\frac{\pi nh}{w}\right)}\frac{1}{\varepsilon_o}\frac{2}{\pi n}\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}}dx'\eta(x')\sin\left(\frac{\pi nx'}{w}\right). \qquad (1.7.8)$$

In principle, this is an **integral equation** which can be solved numerically since the potential on the strip is constant and it equals V_0; many source solution.

At this point we shall employ our *second assumption* namely that the charge is *uniform* across the strip and determine an approximate solution. The first step is to average over the strip region, $|x-w/2|\le\Delta/2$. The left hand side is by definition constant thus



$$V_o = \frac{1}{\Delta}\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dx\,\phi(x, y=h)$$

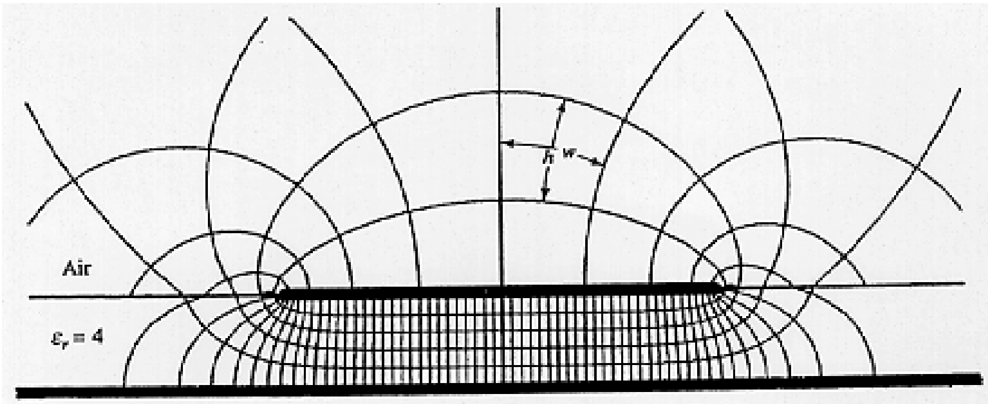
$$= \sum_n \frac{(1/\varepsilon_o)(2/\pi n)}{1+\varepsilon_r \operatorname{ctanh}\left(\frac{\pi n h}{w}\right)} \left[\frac{1}{\Delta}\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dx \sin\left(\frac{\pi n x}{w}\right)\right]\left[\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dx'\eta(x')\sin\left(\frac{\pi n x'}{w}\right)\right].$$

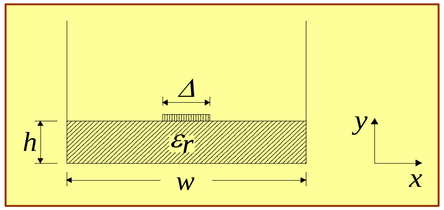
(1.7.9)

Explicitly our assumption that the charge is uniformly distributed, implies $\eta \simeq Q/\Delta_z\Delta$, therefore

$$V_o = \frac{Q}{\Delta_z}\frac{1}{\varepsilon_0}\sum_n \frac{2}{\pi n}\frac{1}{1+\varepsilon_r \operatorname{ctanh}\left(\frac{\pi n h}{w}\right)}\left[\frac{1}{\Delta}\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dx \sin\left(\frac{\pi n x}{w}\right)\right]^2$$

and finally the capacitance per unit length is





$$C = \frac{Q/\Delta_z}{V_o} \simeq \frac{\varepsilon_o \pi/2}{\sum_n \frac{1/n}{1+\varepsilon_r \operatorname{ctanh}\left(\frac{\pi n h}{w}\right)} \sin^2\left(\frac{\pi n}{2}\right) \operatorname{sinc}^2\left(\frac{\pi n \Delta}{2w}\right)} \tag{1.7.10}$$

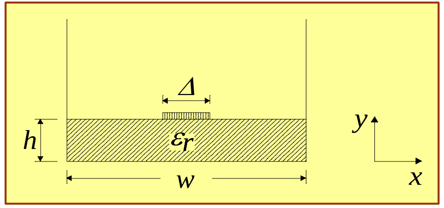
With this expression we can, in principle calculate all the parameters of the microstrip. For evaluation of the **inductance per unit length** we use the fact that the dielectric material cannot have any impact on the DC inductance. Moreover, we know that in the absence of the dielectric ($\varepsilon_r = 1$), the propagation number is ω / c and the characteristic impedance satisfies

$$Z_c = \frac{1}{C(\varepsilon_r = 1)} \frac{1}{c} \sqrt{1} = \sqrt{\frac{L(\varepsilon_r = 1)}{C(\varepsilon_r = 1)}}. \tag{1.7.11}$$

Since the DC magnetic field is totally independent of the dielectric coefficient of the medium (electric property), we deduce from the expression of above that

$$\Rightarrow \qquad L(\varepsilon_r = 1) = \frac{1}{c^2 C(\varepsilon_r = 1)}, \tag{1.7.12}$$

or explicitly



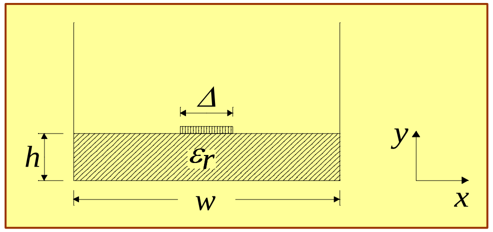
$$L=\mu_o\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{2\nu+1}\left\{\exp\left[-\pi(2\nu+1)\frac{h}{w}\right]\sinh\left[\pi(2\nu+1)\frac{h}{w}\right]\right\}\mathrm{sinc}^2\left[\frac{\pi}{2}\frac{\Delta}{w}(2\nu+1)\right]. \quad (1.7.13)$$

With the last expression and (1.7.10) we can calculate the characteristic impedance of the microstrip

$$Z_c=\sqrt{\frac{L}{C}}=\eta_o\frac{2}{\pi}\left[\sum_{\nu=0}^{\infty}\frac{e^{-h_\nu}\sinh(h_\nu)\mathrm{sinc}^2(\Delta_\nu)}{2\nu+1}\right]^{1/2}\left\{\sum_{\nu=0}^{\infty}\frac{\mathrm{sinc}^2(\Delta_\nu)}{(2\nu+1)[1+\varepsilon_r\mathrm{ctanh}(h_\nu)]}\right\}^{1/2}, \quad (1.7.14)$$

where $h_\nu\equiv\pi(2\nu+1)h/w$ and $\Delta_\nu\equiv\frac{\pi}{2}\frac{\Delta}{w}(2\nu+1)$. The next parameter that remains to be determined is the phase velocity. Since L and C are known, we know that $\beta=\omega\sqrt{LC}$ implying that

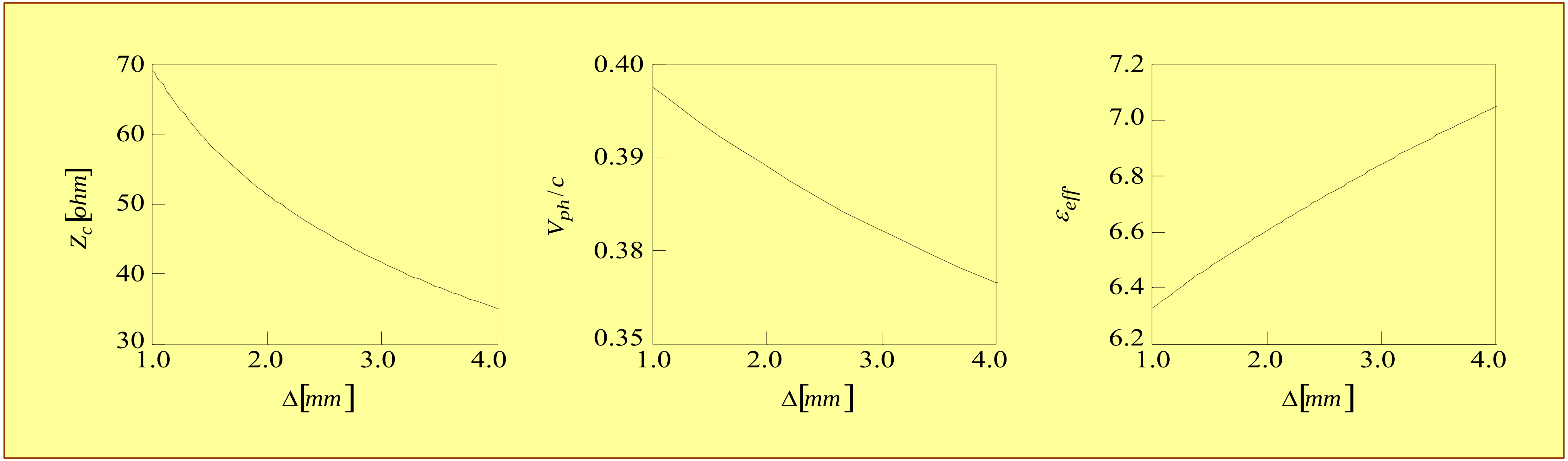
$$V_{ph}=\frac{1}{\sqrt{LC}}=c\sqrt{\frac{\sum_{\nu=0}^{\infty}\frac{\mathrm{sinc}^2(\Delta_\nu)}{2\nu+1}\frac{1}{1+\varepsilon_r\mathrm{ctanh}(h_\nu)}}{\sum_{\nu=0}^{\infty}\frac{\mathrm{sinc}^2(\Delta_\nu)}{2\nu+1}\exp(-h_\nu)\sinh(h_\nu)}}. \quad (1.7.15)$$



Contrary to cases encountered so far the dielectric material fills only **part** of the entire volume. As a result, only part of the electromagnetic field experiences the dielectric. It is therefore natural to determine the **effective dielectric coefficient** experienced by the field. This quantity may be defined in several ways. One possibility is to use the fact that when the dielectric fills the entire space we have the phase velocity in $V_{ph} = \frac{c}{\sqrt{\varepsilon_r}}$ it becomes natural to define the effective dielectric coefficient as $\varepsilon_{\rm eff} \equiv \frac{c^2}{V_{\rm ph}^2}$ thus

$$\varepsilon_{\rm eff} = \frac{\sum_{\nu=0}^{\infty} \frac{\operatorname{sinc}^2(\Delta_\nu)}{2\nu+1} \exp(-h_\nu)\sinh(h_\nu)}{\sum_{\nu=0}^{\infty} \frac{\operatorname{sinc}^2(\Delta_\nu)}{2\nu+1} \frac{1}{1+\varepsilon_r \operatorname{ctanh}(h_\nu)}}. \qquad (1.7.16)$$

The following figures illustrate the dependence of the various parameters on the geometric parameters.

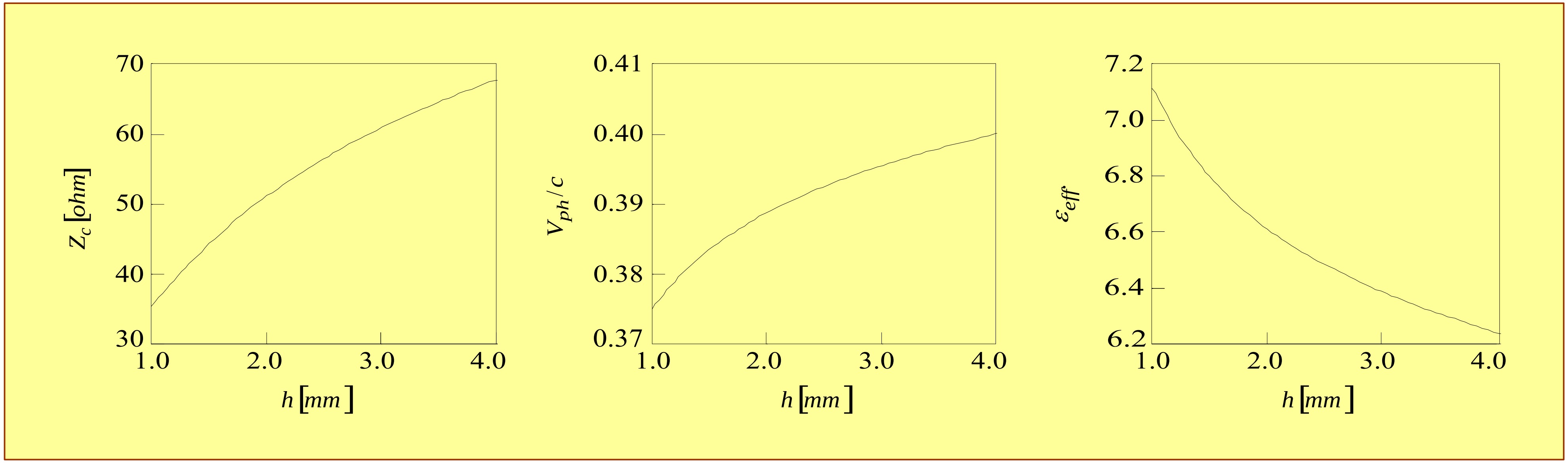


(a) (b) (c)

(a) Characteristic impedance vs. Δ; $h = 2\text{mm}, w = 20\text{mm}, \varepsilon_r = 10$ and $\nu < 100$.

(b) Phase velocity vs. Δ; $h = 2\text{mm}$, $w = 20\text{mm}$, $\varepsilon_r = 10$ and $\nu < 100$.

(c) Effective dielectric coefficient vs. Δ; $h = 2\text{mm}$, $w = 20\text{mm}$, $\varepsilon_r = 10$ and $\nu < 100$.



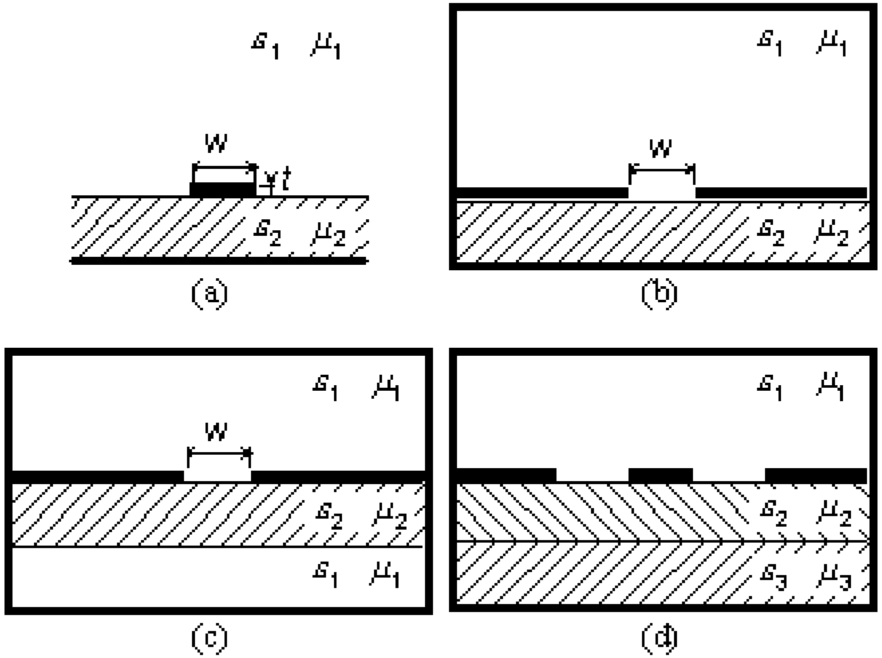
(a) (b) (c)

(a) Characteristic impedance vs. the height h; $\Delta = 2$ mm, $w = 20$ mm, $\varepsilon_r = 10$, $\nu < 100$.

(b) Phase velocity vs. the height h; $\Delta = 2$ mm , $w = 20$ mm, $\varepsilon_r = 10$, $\nu < 100$.

(c) Effective dielectric coefficient vs. the height h; $\Delta = 2$ mm, $w = 20$ mm, $\varepsilon_r = 10$, $\nu < 100$.

Finally the figure below shows several alternative configurations



Exercise 1.8: Determine the effective dielectric coefficient relying on energy confinement.

Exercise 1.9: What fraction of the energy is confined in the dielectric and how the various parameters affect this fraction?

Exercise 1.10: Examine the effect of the dielectric coefficient on Z_c, ε_{eff} and V_{ph}. Compare with the case where the dielectric fills the entire space. (For solution see Appendix 11.1)

Exercise 1.11: Show that if $w \gg h, \Delta$ the various quantities are independent of w. Explain!! (For solution see Appendix 11.2)

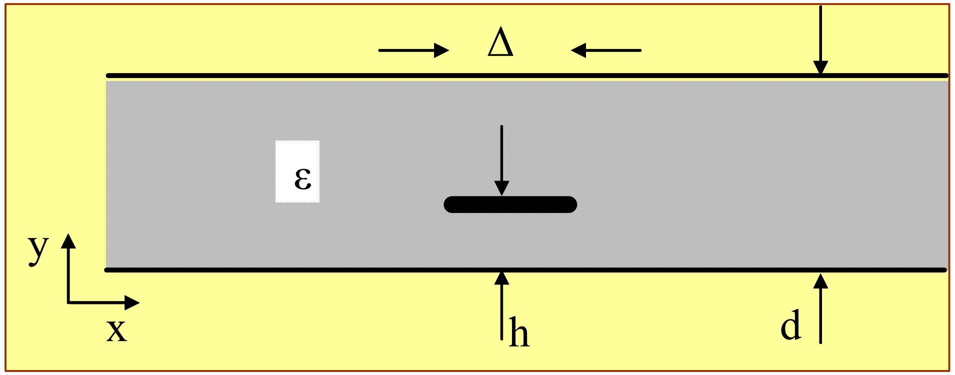
Exercise 1.12: Analyze the effect of dielectric and permeability loss on a micro-strip.

Exercise 1.13: Calculate the ohmic loss. Analyze the effect of the strip and ground separately.

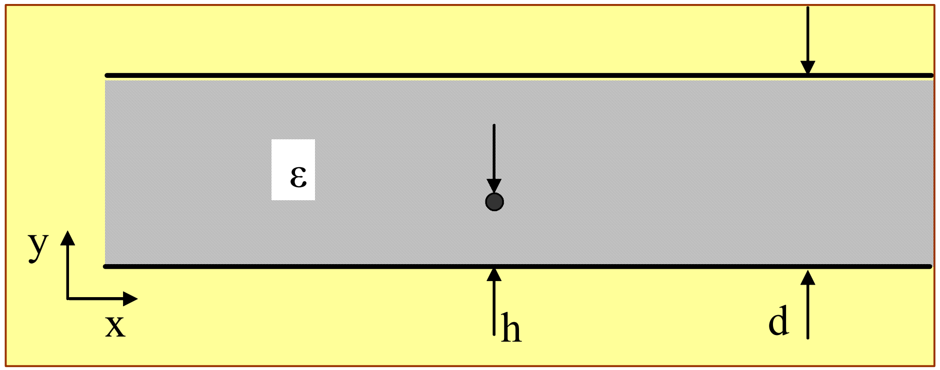
Exercise 1.14: Determine the effect of the edges on the electric parameters (L, C).

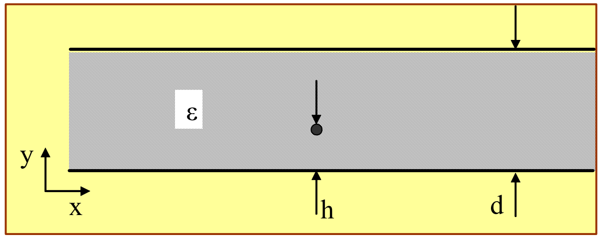
1.8 Stripline

Being open on the top side, the microstrip has limited ability to confine the electromagnetic field. For this reason we shall examine now the stripline which has a metallic surface on its top. The basic configuration of a stripline is illustrated below



The model we shall utilize first replaces the central strip with a wire as illustrated below and as in Section 1.7 our goal is to calculate the parameters of the line.





For evaluation of the capacitance per unit length it is first assumed that the charge density is given by

$$\rho(x,y) = \frac{Q}{\Delta_z}\delta(x)\delta(y-h), \tag{1.8.1}$$

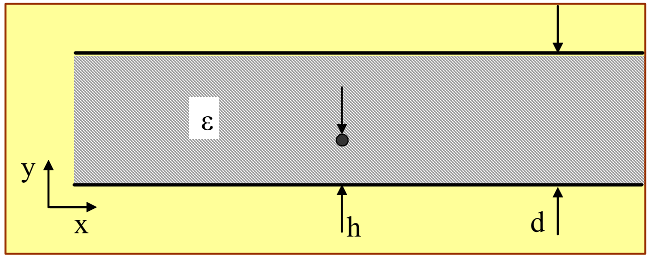
and we need to solve the Poisson equation subject to trivial boundary conditions on the two electrodes. Thus

$$\left.\begin{array}{l} \nabla_t^2\phi = -\dfrac{\rho}{\varepsilon_o\varepsilon_r} \\ \phi(y=0)=0 \\ \phi(y=d)=0 \end{array}\right\} \Rightarrow \phi(x,y) = \sum_n \phi_n(x)\sin\left(\frac{\pi n y}{d}\right). \tag{1.8.2}$$

Substituting the expression in the right hand side in the Poisson equation we have

$$\sum_n \left[\frac{d^2}{dx^2}\phi_n(x) - \left(\frac{\pi n}{d}\right)^2 \phi_n(x)\right]\sin\left(\frac{\pi n y}{d}\right) = \frac{-Q}{\varepsilon_o\varepsilon_r\Delta_z}\delta(x)\delta(y-h) \tag{1.8.3}$$

and the orthogonality of the trigonometric function we obtain



$$\left[\frac{d^2}{dx^2}-\left(\frac{\pi n}{d}\right)^2\right]\phi_n(x)=\frac{-Q}{\varepsilon_o\varepsilon_r\Delta_z}\delta(x)\frac{2}{d}\sin\left(\frac{\pi nh}{d}\right)=-Q_n\delta(x), \tag{1.8.4}$$

where

$$Q_n \equiv \frac{2Q}{\varepsilon_o\varepsilon_r d\Delta_z}\sin\left(\frac{\pi nh}{d}\right).$$

The solution of (1.8.4) is given by

$$\phi_n(x)=\begin{cases} A_n \exp\left(-\pi n\dfrac{x}{d}\right) & x>0 \\ B_n \exp\left(\pi n\dfrac{x}{d}\right) & x<0 \end{cases} \tag{1.8.5}$$

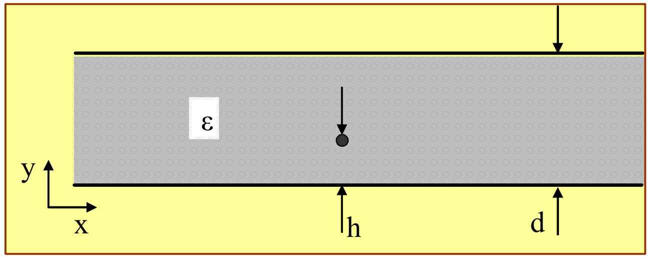
and since the potential has to be continuous at $x=0$ then

$$A_n = B_n, \tag{1.8.6}$$

integration of (1.8.4) determines the discontinuity:

$$\left.\frac{d\phi_n}{dx}\right|_{x=0^+}-\left.\frac{d\phi_n}{dx}\right|_{x=0^-}=-Q_n \quad\Rightarrow\quad -\left(\frac{\pi n}{d}\right)[A_n+B_n]=-Q_n. \tag{1.8.7}$$

From (1.8.6) and (1.8.7) we find



$$A_n = B_n = \frac{Q_n}{2\left(\frac{\pi n}{d}\right)} = \frac{Q}{\varepsilon_o \varepsilon_r \Delta_z} \frac{h}{d} \operatorname{sinc}\left(\pi n \frac{h}{d}\right). \tag{1.8.8}$$

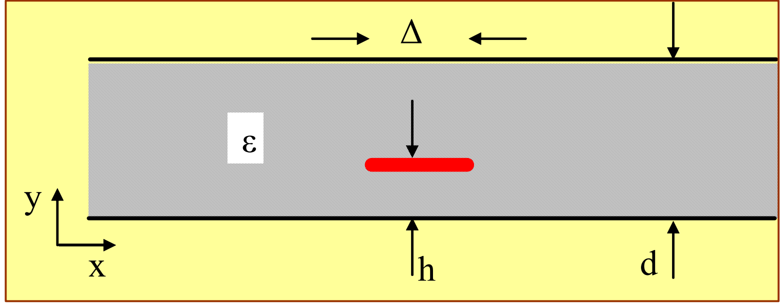
This result permits us to write the solution of the potential in the entire space as

$$\phi(x,y) = \begin{cases} \dfrac{Q}{\varepsilon_o \varepsilon_r \Delta_z} \dfrac{h}{d} \sum_{n=1}^{\infty} \operatorname{sinc}\left(\dfrac{\pi n h}{d}\right) \sin\left(\dfrac{\pi n y}{d}\right) e^{-\frac{\pi n x}{d}} & x \ge 0 \\ \dfrac{Q}{\varepsilon_o \varepsilon_r \Delta_z} \dfrac{h}{d} \sum_{n=1}^{\infty} \operatorname{sinc}\left(\dfrac{\pi n h}{d}\right) \sin\left(\dfrac{\pi n y}{d}\right) e^{+\frac{\pi n x}{d}} & x \le 0. \end{cases} \tag{1.8.9}$$

At this stage we can return to the initial configuration and assume that the central strip is a superposition of charges Q_i located at x_i and since the system is linear, we apply the superposition principle thus

$$\phi(x,y) = \frac{h/d}{\varepsilon_o \varepsilon_r \Delta_z} \sum_n \operatorname{sinc}\left(\frac{\pi n h}{d}\right) \sin\left(\frac{\pi n y}{d}\right) \sum_i Q_i \exp\left[-\pi n \frac{|x - x_i|}{d}\right]. \tag{1.8.10}$$

In the case of a continuous distribution we should replace



$$\sum_i Q_i e^{-\frac{\pi n}{d}|x - x_i|} = \frac{1}{\Delta}\int dx' Q(x') e^{-\frac{\pi n}{d}|x-x'|} \qquad (1.8.11)$$

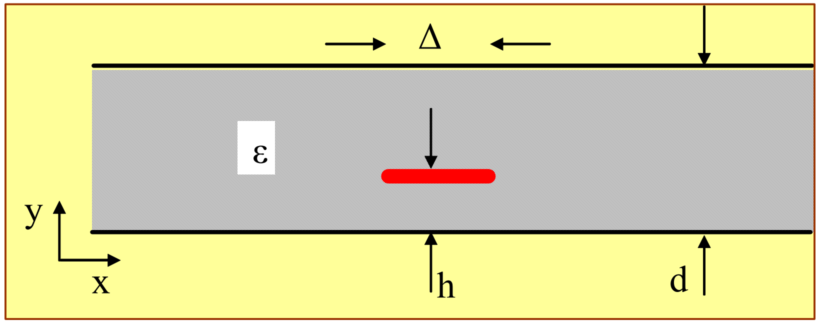
and consequently

$$\phi(x,y) = \frac{h/d}{\varepsilon_o \varepsilon_r \Delta_z}\sum_n \text{sinc}\left(\frac{\pi n h}{d}\right)\sin\left(\frac{\pi n y}{d}\right)\frac{1}{\Delta}\int dx' Q(x')\exp\left(-\frac{\pi n}{d}|x-x'|\right), \qquad (1.8.12)$$

which again leads us to an integral equation; note that the surface changes density as $\rho_s(x) = Q(x)/\Delta\Delta_z$. As in the microstrip case, we shall assume *uniform* distribution therefore

$$\phi(x,y) = \frac{h/d}{\varepsilon_o \varepsilon_r \Delta_z}\sum_n \text{sinc}\left(\frac{\pi n h}{d}\right)\sin\left(\frac{\pi n y}{d}\right)\frac{Q_o}{\Delta}\int_{-\Delta/2}^{\Delta/2} dx' \exp\left(-\frac{\pi n}{d}|x-x'|\right). \qquad (1.8.13)$$

The potential is constant on the strip



$$V_o = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} dx\,\phi(x, y=h)$$

$$= \frac{(h/d)Q_o}{\varepsilon_o\varepsilon_r\Delta_z}\sum_{n=1}^{\infty}\left(\frac{\pi nh}{d}\right)^{-1}\sin^2\left(\frac{\pi nh}{d}\right)\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} dx\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} dx'\exp\left(-\frac{\pi n}{d}|x-x'|\right) \quad (1.8.14)$$

and the two integrals may be simplified to read

$$\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} dx\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} dx'\exp\left(-\frac{\pi n}{d}|x-x'|\right) = \left(\frac{\pi n}{2}\frac{\Delta}{d}\right)^{-1}\left[1-\exp\left(-\frac{\pi n}{2}\frac{\Delta}{d}\right)\mathrm{sinhc}\left(\frac{\pi n\Delta}{2d}\right)\right] \quad (1.8.15)$$

such that

$$V_o = \frac{(h/d)Q_o\left(\frac{2h}{\Delta}\right)}{\varepsilon_o\varepsilon_r\Delta_z}\sum_{n=1}^{\infty}\mathrm{sinc}^2\left(\frac{\pi nh}{d}\right)\left[1-\exp\left(-\frac{\pi n\Delta}{2d}\right)\mathrm{sinhc}\left(\frac{\pi n\Delta}{2d}\right)\right]. \quad (1.8.16)$$

The last result enables us to write the following expression for the capacitance per unit length

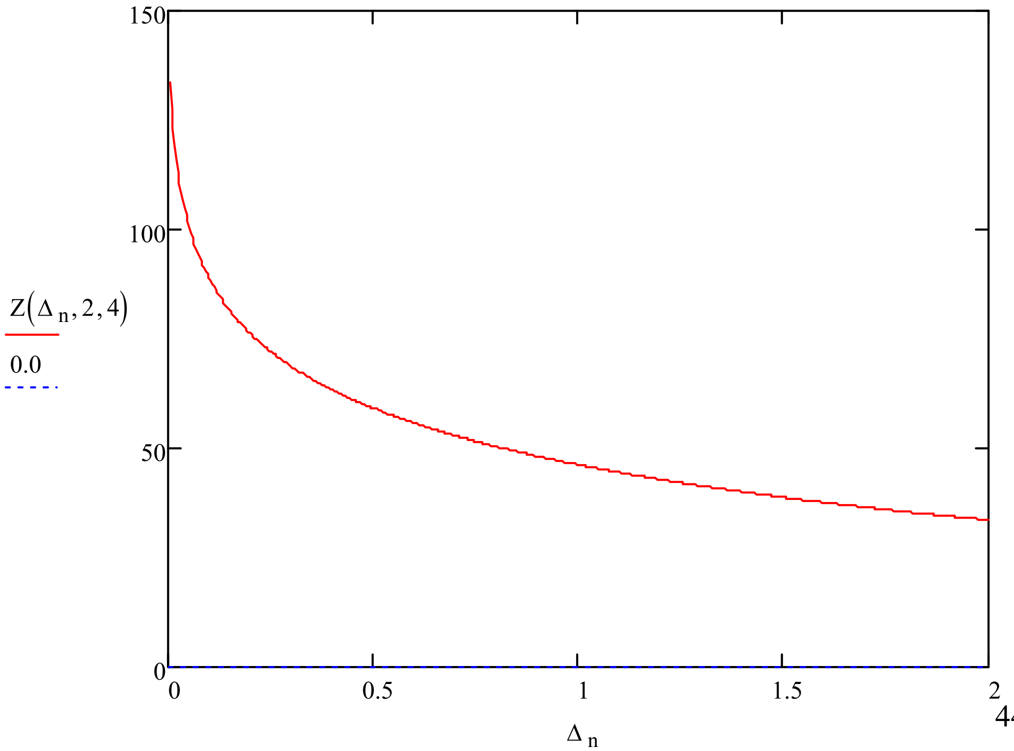
$$C = \frac{Q_o/\Delta_z}{V_o} = \varepsilon_o\varepsilon_r\frac{1}{2}\left(\frac{\Delta d}{h^2}\right)\left\{\sum_n \mathrm{sinc}^2\left(\frac{\pi nh}{d}\right)\left[1-\exp(-\xi_n)\mathrm{sinhc}(\xi_n)\right]\right\}^{-1} \quad (1.8.17)$$

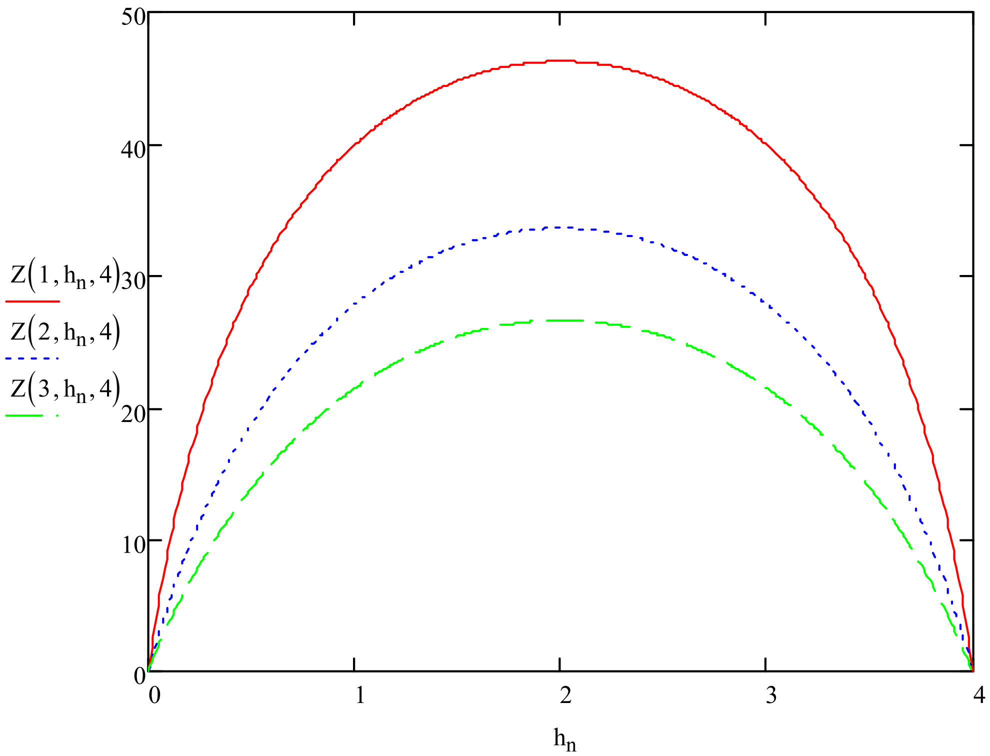
whereas $\xi_n = \pi n\Delta / 2d$, thus with it the characteristic impedance reads

$$Z_c = \frac{1}{CV_{ph}} = \eta_o \frac{2}{\sqrt{\varepsilon_r}} \frac{h^2}{\Delta d} \sum_n \mathrm{sinc}^2\left(\frac{\pi n h}{d}\right)\left[1 - \exp\left(-\xi_n\right)\mathrm{sinhc}(\xi_n)\right]. \qquad (1.8.18)$$

The two frames show the impedance dependence on the width and height of the strip $\varepsilon = 10$

$$Z(\Delta, h, d) := 377 \cdot \frac{2}{\sqrt{\varepsilon}} \cdot \frac{h^2}{\Delta \cdot d} \cdot \left[\sum_n \left[\mathrm{sinc}\left(n \cdot \pi \frac{h}{d}\right)^2 \cdot \left(1 - \exp\left(-\pi \cdot n \cdot \frac{\Delta}{2d}\right) \mathrm{sinhc}\left(\pi \cdot n \cdot \frac{\Delta}{2d}\right)\right)\right]\right]$$





Exercise 1.15: Determine the inductivity per unit length and analyse the dependence of the various characteristics on the geometric parameters. (For solution see Appendix 11.3)

Exercise 1.16: Compare the dependence of the various characteristics of the stripline and microstrip as a function of the geometric parameters.

Exercise 1.17: Compare micro-strip and strip-line from the perspective of sensitivity to the dielectric coefficient. (For solution see Appendix 11.4)

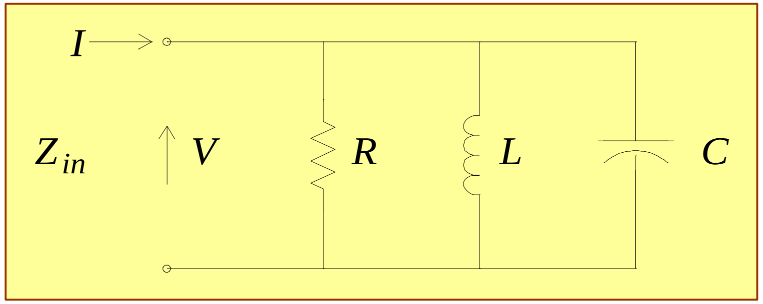
Exercise 1.18: Determine the error associated with the assumption that the charge is *uniform* across the strip.

Exercise 1.19: Analyze the effect of a strip of finite thickness. Remember that throughout this calculation the strip was assumed to have a negligible thickness.

1.9 Resonator Based on Transmission Line

1.9.1 Short Recapitulation

Resonant circuits are of great importance for oscillator circuits, tuned amplifiers, frequency filter networks, wavemeters for measuring frequency. Electric resonant circuits have many features in common, and it will be worthwhile to review some of these by using a conventional lumped-parameter RLC parallel network as an example, the Figure illustrates a typical low-frequency resonant circuit. The resistance R is usually only an equivalent resistance that accounts for the power loss in the inductor L and capacitor C as well as the power extracted from the resonant system by some external load coupled to the resonant circuit. One possible definition of resonance relies on the fact that at resonance the input impedance is pure real and equal to R implying



$$Z_{in} = \frac{P_l + 2j\omega(W_m - W_e)}{II^*/2}.$$

Although this equation is valid for a one-port circuit, resonance always occurs when $W_m = W_e$, if we define resonance to be that condition which corresponds to a pure resistive input impedance or explicitly $\omega_0 = 1/\sqrt{LC}$; note that these are the lumped capacitance (C) and inductance (L).

An important parameter specifying the frequency selectivity, and performance in general, of a resonant circuit is the quality factor, or Q. A very general definition of Q that is applicable to all resonant ($W_e = W_m$) systems is

$$Q = \frac{\omega_0(\text{time} - \text{average energy stored in the system})}{\text{energy loss per second in the system}}. \tag{1.9.2}$$

hence,

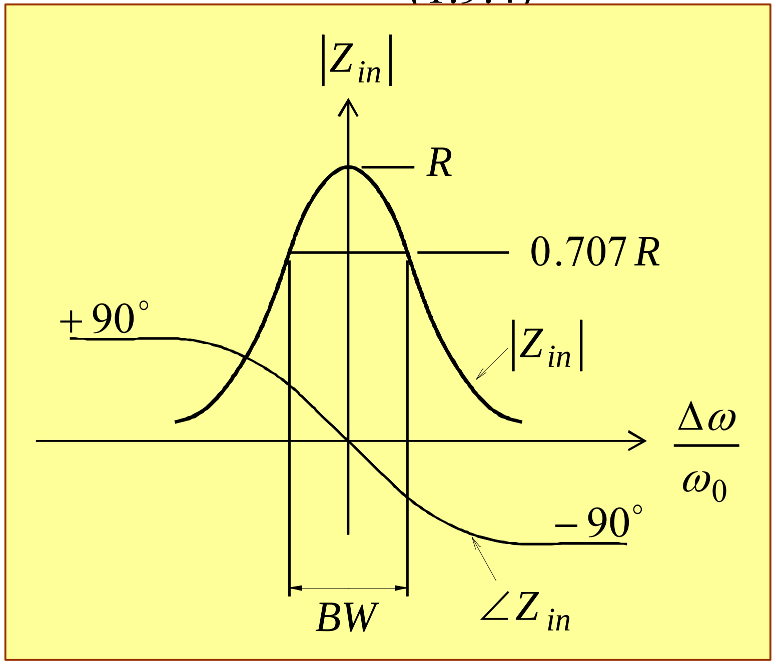
$$Q = \omega_0 RC = R/\omega_0 L. \tag{1.9.3}$$

In the vicinity of resonance, say $\omega = \omega_0 + \Delta\omega$, the input impedance can be expressed in a relatively simple form. We have

$$Z_{in} = \frac{\omega_0^2 RL}{\omega_0^2 L + j2R\Delta\omega} = \frac{R}{1 + j2Q(\Delta\omega/\omega_0)}. \tag{1.9.4}$$

A plot of Z_{in} as a function of $\Delta\omega/\omega_0$ is given below. When $|Z_{in}|$ has fallen to $1/\sqrt{2}$ (half the power) of its maximum value, its phase is 45° if $\omega < \omega_0$ and -45° if $\omega > \omega_0$ thus

$$2Q\frac{\Delta\omega}{\omega_0} = 1 \Rightarrow \boxed{\Delta\omega = \frac{\omega_0}{2Q}} \tag{1.9.5}$$



The fractional bandwidth BW between the $0.707R$ points is twice this value, hence

$$Q = \frac{\omega_0}{2\Delta\omega} = \frac{1}{BW}. \tag{1.9.6}$$

If the resistor R in Fig. 8 represents the loss in the resonant circuit only, the Q give by (1.9.3) is called the **unloaded** Q. If the resonant circuit is coupled to an external load that absorbs a certain amount of power, this loading effect can be represented by an additional resistor R_L in parallel with R. The total resistance is now less, and consequently the new Q is also smaller. The Q, called the loaded Q and denoted Q_L, is

$$Q_L = \frac{RR_L/(R+R_L)}{\omega_0 L}. \tag{1.9.7}$$

The external Q, denoted Q_e, is defined to be the Q that would result if the *resonant circuit were loss-free* and only the loading by the external load was present. Thus

$$Q_e = \frac{R_L}{\omega_0 L} \tag{1.9.8}$$

leading to

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q}. \tag{1.9.9}$$

Another parameter of importance in connection with a resonant circuit is the decay factor τ. This parameter measures the rate at which the oscillations would decay if the driving source were removed. Significantly, with losses present, the energy stored in the resonant circuit will decay at a rate proportional to the average energy present at any time (since $P_l \propto VV^*$ and $W \propto VV^*$, we have $P_l \propto W$), so that

$$\frac{dW}{dt} = -\frac{2}{\tau}W \quad \Rightarrow \quad W = W_0 \exp\left(-2\frac{t}{\tau}\right) \tag{1.9.10}$$

where W_0 is the average energy present at $t = 0$. But the rate of decrease of W must equal the power loss, so that

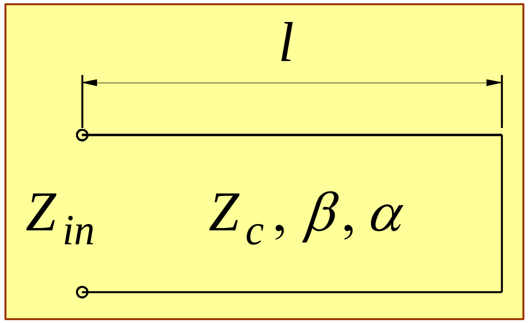
$$-\frac{dW}{dt} = \frac{2}{\tau}W = P_l$$

and consequently,

$$\frac{1}{\tau} = \frac{P_l}{2W} = \frac{\omega_0}{2}\frac{P_l}{\omega_0 W} = \frac{\omega_0}{2Q}. \tag{1.9.11}$$

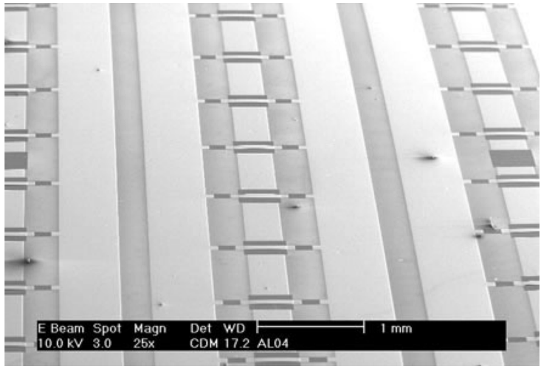
Thus, the decay factor is proportional to the Q. In place of (1.9.10) we now have

$$W = W_0 \exp\left(-\frac{\omega_0}{Q}t\right). \tag{1.9.12}$$



1.9.2 Short-Circuited Line

By analogy to the previous section, consider a short-circuited line of length l, parameters R, L, C per unit length, as in Fig. 10. Let $l = \lambda_0 / 2$ at $f = f_0$, that is, at $\omega = \omega_0$. For f near f_0, say $f = f_0 + \Delta f$, $\beta l = 2\pi f l / c$, $\pi\omega / \omega_0 = \pi + \pi\Delta\omega / \omega_0$, since at $\omega_0, \beta l = \pi$. The input impedance is given by

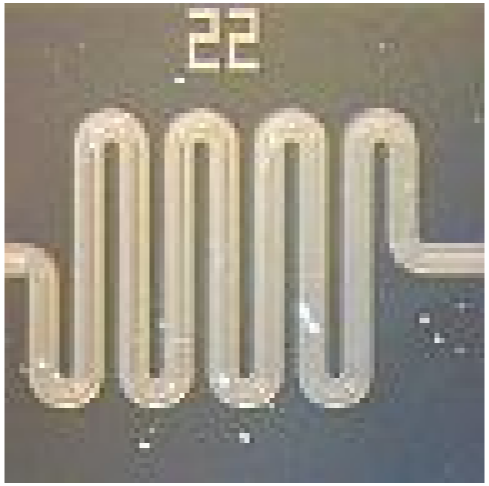


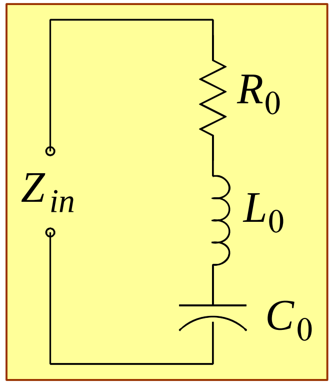
$$Z_{in} = Z_c \tanh(j\beta l + \alpha l) = Z_c \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tan \beta l \tanh \alpha l}. \qquad (1.9.13)$$

But $\tanh \alpha l \simeq \alpha l$ since we are assuming small losses, so that $\alpha l \ll 1$. Also $\tan \beta l = \tan(\pi + \pi\Delta\omega / \omega_0) = \tan \pi\Delta\omega / \omega_0 \simeq \pi\Delta\omega / \omega_0$ since $\Delta\omega / \omega_0$ is small. Hence

$$Z_{in} = Z_c \frac{\alpha l + j\pi\Delta\omega / \omega_0}{1 + j\alpha l \pi\Delta\omega / \omega_0} \approx Z_c \left(\alpha l + j\pi \frac{\Delta\omega}{\omega_0} \right) \qquad (1.9.14)$$

since the second term in the denominator is very small. Now $Z_c = \sqrt{L / C}$, $\alpha = \frac{1}{2} R Y_c = (R / 2)\sqrt{C / L}$, and $\beta l = \omega_0 \sqrt{LC} l = \pi$; so $\pi / \omega_0 = l\sqrt{LC}$, and the expression for Z_{in} becomes





$$Z_{in} = \sqrt{\frac{L}{C}}\left(\frac{l}{2}R\sqrt{\frac{C}{L}} + j\Delta\omega l\sqrt{LC}\right) = \frac{1}{2}Rl + jlL\Delta\omega. \qquad (1.9.15)$$

It is of interest to compare (1.9.15) with a series $R_0L_0C_0$ circuit illustrated above. For this circuit

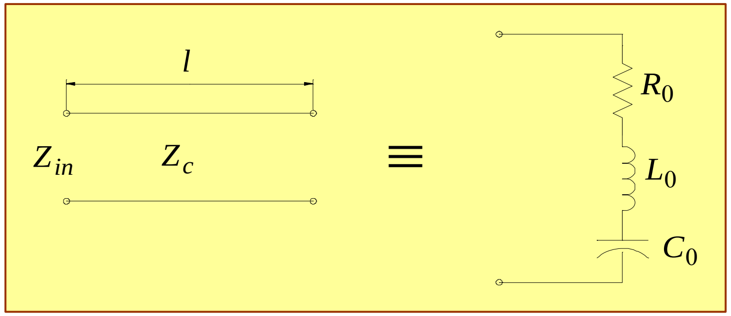
$$Z_{in} = R_0 + j\omega L_0\left(1 - 1/\omega^2 L_0 C_0\right).$$

If we let $\omega_0^2 = 1/L_0C_0$, then $Z_{\text{in}} = R_0 + j\omega L_0\left(\omega^2 - \omega_0^2\right)/\omega^2$. Now if $\omega - \omega_0 = \Delta\omega$ is small then

$$Z_{in} \simeq R_0 + 2jL_0\Delta\omega. \qquad (1.9.16)$$

By comparison with (1.9.15), we see that in the vicinity of the frequency for which $l = \lambda_0/2$, the short-circuited line behaves as a series resonant circuit with resistance $R_0 = Rl/2$ and inductance $L_0 = Ll/2$. We note that Rl, Ll are the total resistance and inductance of the line; so we might wonder why the factors $1/2$ arise: recall that the current on the short-circuited line is half sinusoid, and hence the effective circuit parameters R_0, L_0 are only one-half of the total line quantities.The Q of the short-circuited line may be defined as for the circuit

$$Q = \frac{\omega_0 L_0}{R_0} = \frac{\omega_0 L}{R} = \frac{\beta}{2\alpha}. \qquad (1.9.17)$$

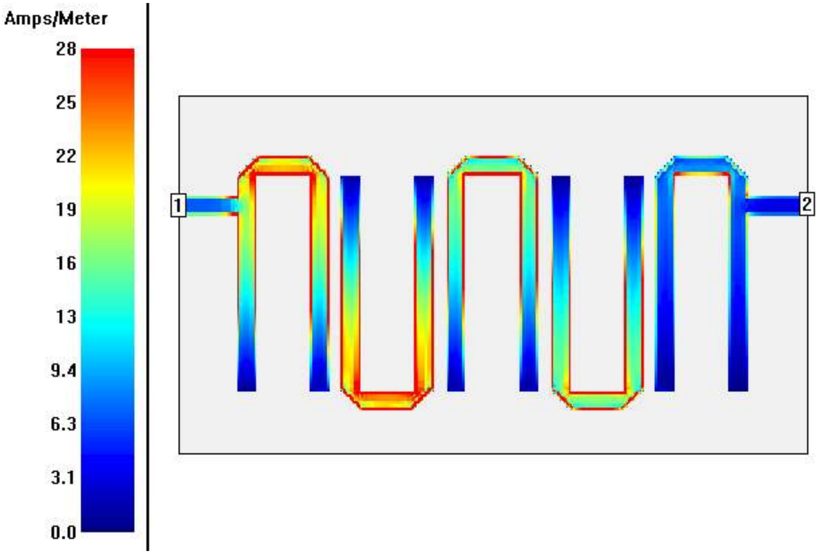


1.9.3 Open-Circuited Line

By means of an analysis similar to that used earlier, it is readily verified that an open-circuited transmission line is equivalent to a series resonant circuit in the vicinity of the frequency for which it is an odd multiple of a quarter wavelength long. The equivalent relations are

$$
\begin{aligned}
&Z_{in} \simeq \left[\alpha l + j(\pi/2)(\Delta\omega/\omega_0)\right] Z_c = Rl/2 + j\Delta\omega L l \\
&l = \lambda_0/4,\ R_0 = Rl/2,\ L_0 = Ll/2,\ \omega_0^2 = 1/L_0 C_0
\end{aligned}
\qquad (1.9.18)
$$

Comment: Note that formally from (1.9.13) in the lossless case i.e., $Z_{in} = jZ_c \tan\beta l$ we conclude that there are many (infinite) resonances since $\tan(\beta l)$ vanishes for $\beta l = \pi$ but also for $\beta l = \pi n,\ \ n = 1,2,3\ldots$ corresponding to all the "series" resonances. In case of "parallel" resonances the condition $\tan\beta l \simeq \pm\infty$ is satisfied for $\beta l = \dfrac{\pi}{2} + \pi n,\ \ n = 1,2,\ldots.$ In practice, only the first resonance is used since beyond that the validity of the approximations leading to the equations are questionable.



1.10 Pulse Propagation

1.10.1 Semi-Infinite Structure

So far the discussion has focused on solution of problems in the frequency domain. In this section we shall discuss some **time-domain** features. Let us assume that at the input of a semi-infinite and lossless transmission line we know the voltage pulse $V(z=0,t)=V_0(t)$. In general in the absence of reflections

$$V(z,t)=\int d\omega\bar{V}(\omega)e^{j\omega t-j\beta(\omega)z} \qquad (1.10.1)$$

and specifically

$$V(z=0,t)=\int d\omega\bar{V}(\omega)e^{j\omega t} \qquad (1.10.2)$$

or explicitly, **the voltage spectrum** $\bar{V}(\omega)$ is the **Fourier transform of the input voltage**

$$\bar{V}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dtV(z=0,t)e^{-j\omega t}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dtV_0(t)e^{-j\omega t}. \qquad (1.10.3)$$

Consequently, substituting in Eq.(1.10.1) we get

$$\begin{aligned}V(z,t)&=\int_{-\infty}^{\infty}d\omega e^{j\omega t-j\beta(\omega)z}\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'V(z=0,t')e^{-j\omega t'}\\&=\int_{-\infty}^{\infty}dt'V(z=0,t')\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{j\omega(t-t')-j\beta(\omega)z}\end{aligned} \qquad (1.10.4)$$

and in the case of a **dispersionless line** we have $\beta(\omega) = \frac{\omega}{c}\sqrt{\varepsilon_r}$ which leads to

$$V(z,t) = \int_{-\infty}^{\infty} dt' V(z=0,t')\delta\left(t'-t+\frac{z}{c}\sqrt{\varepsilon_r}\right) = V\left(z=0, t'=t-\frac{z}{c}\sqrt{\varepsilon_r}\right) \quad (1.10.5)$$

implying that the **pulse shape is preserved** as it propagates in the z-direction. If the phase velocity is frequency-dependent, then different frequencies propagate at *different velocities* and the shape of the pulse is not preserved. As a simple example let us assume that the transmission line is filled with gas

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \quad (1.10.6)$$

Since

$$V(z,t) = \int_{-\infty}^{\infty} dt' V\left(z=0,t'\right)\frac{1}{2\pi}\int d\omega \exp\left[j\omega(t-t') - \frac{z}{c}\sqrt{\omega_p^2-\omega^2}\right] \quad (1.10.7)$$

it is evident that sufficiently far away from the input, the low frequencies $(\omega < \omega_p)$ have no contribution and the system acts as a high-pass filter.

The dispersion process may be used to determine the frequency content of a signal. In order to envision the process let us assume that the spectrum of the signal at the *input* is given by