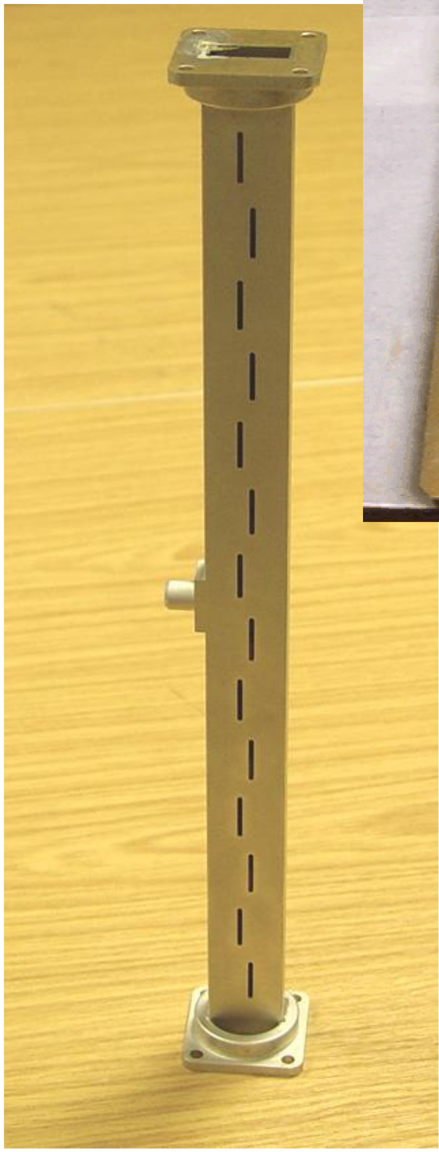
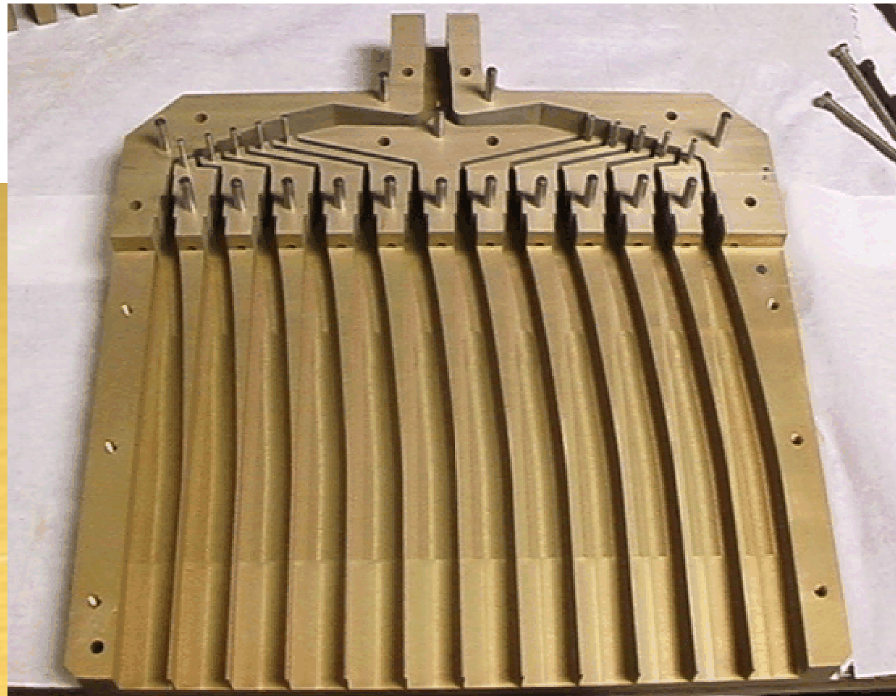
Chapter 2: Waveguides – Fundamentals

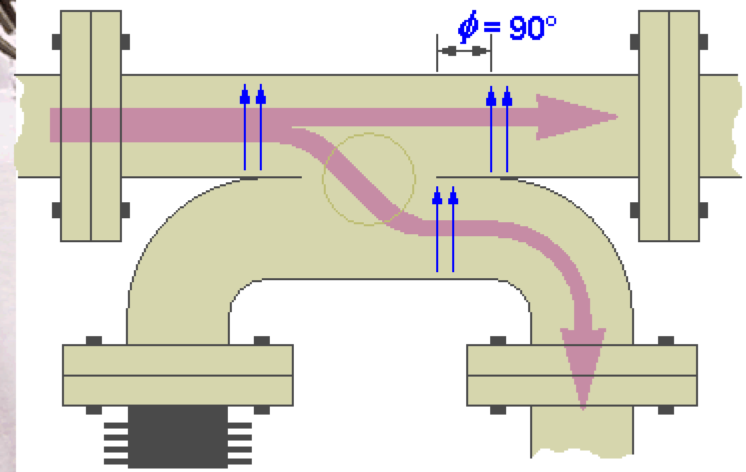
2.1 General Formulation

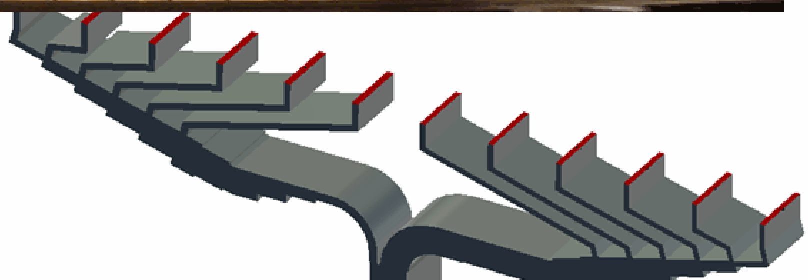
So far we have examined the propagation of electromagnetic waves in a structure consisting of two or more metallic surfaces. This type of structure supports a transverse electromagnetic (TEM) mode. However, if the electromagnetic characteristics of the structure are not uniform across the structure, the mode is not a pure TEM mode but it has a longitudinal field component.

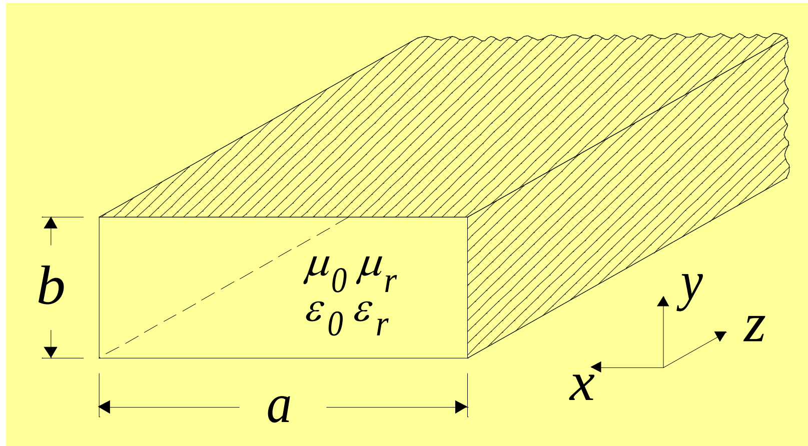
In this chapter we consider the propagation of an electromagnetic wave in a closed metallic structure which is infinite in one direction (z) and it has a rectangular (or cylindrical) cross-section as illustrated in Fig. 1. While the use of this type of waveguide is relatively sparse these days, we shall adopt it since it provides a very convenient mathematical foundation in the form of a set of trigonometric functions. This is an orthogonal set of functions which may be easily manipulated. The approach is valid whenever the transverse dimensions of the structure are comparable with the wavelength.






ϕ = 90°





Rectangular waveguide; a and b are the dimensions of the rectangular cross section. a corresponding to the x coordinate, b to the y coordinate.

The first step in our analysis is to establish the basic ***assumptions*** of our approach:

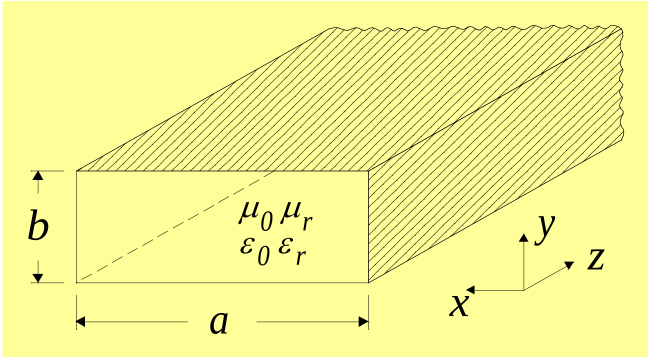
a) The electromagnetic characteristics of the medium: $\mu = \mu_0 \mu_r$ and $\varepsilon = \varepsilon_0 \varepsilon_r$.

b) Steady state operation of the type $\exp(j\omega t)$.

c) No sources in the pipe.

d) Propagation in the z direction -- $\exp(-jk_z z)$; k_z can be either real or imaginary or complex number.

e) The conductivity (σ) of the metal is assumed to be arbitrary large $(\sigma \to \infty)$.



Subject to these assumptions Maxwell's Equations may be written in the following form

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E} \tag{2.1.1}$$

Substituting one equation into the other we obtain the wave equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -j\omega\mu(\vec{\nabla} \times \vec{H}) \qquad \vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = j\omega\varepsilon(\vec{\nabla} \times \vec{E})$$

$$\nabla(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E} = -j\omega\mu(j\omega\varepsilon\vec{E}) \qquad \nabla(\vec{\nabla} \cdot \vec{H}) - \nabla^2\vec{H} = j\omega\varepsilon(-j\omega\mu\vec{H})$$

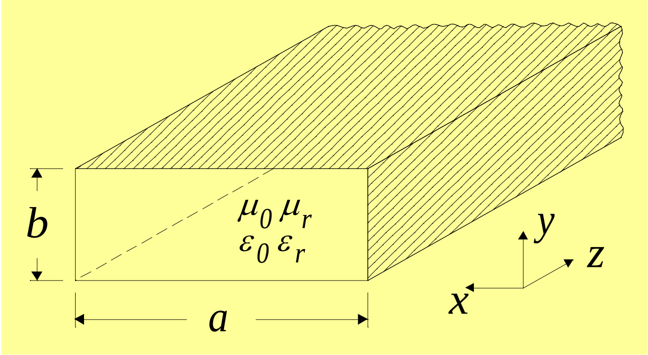
$$\nabla(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E} = \left(\frac{\omega}{\text{v}}\right)^2 \vec{E} \qquad \nabla(\vec{\nabla} \cdot \vec{H}) - \nabla^2\vec{H} = \left(\frac{\omega}{\text{v}}\right)^2 \vec{H} \tag{2.1.2}$$

$$\vec{\nabla} \cdot \varepsilon\vec{E} = 0 \qquad \vec{\nabla} \cdot \mu\vec{H} = 0$$

$$\boxed{\left[\nabla^2 + \left(\frac{\omega}{\text{v}}\right)^2\right]\vec{E} = 0} \qquad \boxed{\left[\nabla^2 + \left(\frac{\omega}{\text{v}}\right)^2\right]\vec{H} = 0}$$

where $\text{v} = 1/\sqrt{\mu\varepsilon}$ is the phase-velocity of a plane wave in the medium. Specifically, we conclude that the z components of the electromagnetic field satisfy

$$\left[\nabla^2 + \frac{\omega^2}{\text{v}^2}\right]E_z = 0, \qquad \left[\nabla^2 + \frac{\omega^2}{\text{v}^2}\right]H_z = 0 \tag{2.1.3}$$



and subject to assumption (d) we have

$$\left(\nabla_\perp^2 - k_z^2 + \frac{\omega^2}{v^2}\right)E_z = 0 \qquad \left(\nabla_\perp^2 - k_z^2 + \frac{\omega^2}{v^2}\right)H_z = 0.$$

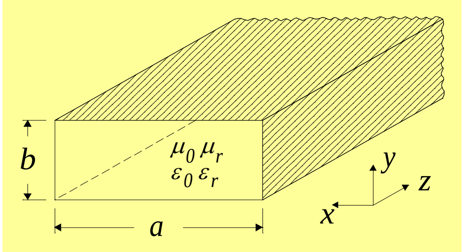
As a second step, it will be shown that assuming the longitudinal components of the electromagentic field are known, the transverse components are readily established. For this purpose we observe that Faraday's Law reads

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \Rightarrow \begin{vmatrix} 1_x & 1_y & 1_z \\ \partial_x & \partial_y & -jk_z \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\vec{H}. \qquad (2.1.4)$$

$$\begin{array}{lllll}
\text{(i)} & 1_x: & \partial_y E_z + jk_z E_y = -j\omega\mu H_x & \rightsquigarrow & jk_z E_y + j\omega\mu H_x = -\partial_y E_z \\
\text{(ii)} & 1_y: & -(\partial_x E_z + jk_z E_x) = -j\omega\mu H_y & \rightsquigarrow & -jk_z E_x + j\omega\mu H_y = \partial_x E_z \\
\text{(iii)} & 1_z: & \partial_x E_y - \partial_y E_x = -j\omega\mu H_z & \rightsquigarrow & \partial_x E_y - \partial_y E_x = -j\omega\mu H_z.
\end{array}$$

In a similar way, Ampere's law reads

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E} \Rightarrow \begin{vmatrix} 1_x & 1_y & 1_z \\ \partial_x & \partial_y & -jk_z \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon\vec{E}, \qquad (2.1.5)$$



or explicitly

$$
\begin{array}{llllll}
\text{(iv)} & 1_x: & \partial_y H_z + jk_z H_y = j\omega\varepsilon E_x & \rightsquigarrow & j\omega\varepsilon E_x - jk_z H_y = & \partial_y H_z \\
\text{(v)} & 1_y: - & (\partial_x H_z + jk_z H_x) = j\omega\varepsilon E_y & \rightsquigarrow & j\omega\varepsilon E_y + jk_z H_x = - & \partial_x H_z \\
\text{(vi)} & 1_z: & \partial_x H_y - \partial_y H_x = j\omega\varepsilon E_z & \rightsquigarrow & \partial_x H_y - \partial_y H_x = & j\omega\varepsilon E_z.
\end{array}
$$

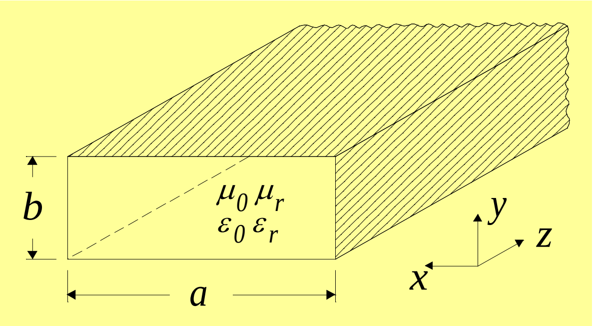
From equations **(ii)** and **(iv)** we obtain

$$
\left.\begin{array}{r} -jk_z E_x + j\omega\mu H_y = \partial_x E_z \\ j\omega\varepsilon E_x - jk_z H_y = \partial_y H_z \end{array}\right\} \rightarrow
\begin{array}{l} H_y = \dfrac{jk_z\omega\varepsilon}{k_z^{\ 2} - (\omega/\mathrm{v})^2}\left[\dfrac{1}{k_z}\partial_x E_z + \dfrac{1}{\omega\varepsilon}\partial_y H_z\right] \\ E_x = \dfrac{j}{k_z^{\ 2} - (\omega/\mathrm{v})^2}\left[k_z\partial_x E_z + \omega\mu\partial_y H_z\right] \end{array}
\tag{2.1.6}
$$

It is convenient at this point to define the **transverse wavenumber**

$$
k_\perp^2 \equiv \frac{\omega^2}{\mathrm{v}^2} - k_z^{\ 2}
\tag{2.1.7}
$$

That as we shall shortly see, has a special physical meaning.



This allows us to write the last two expressions in the following form

$$E_x = \frac{-jk_z}{k_\perp^2}\partial_x E_z - \frac{j\omega\mu}{k_\perp^2}\partial_y H_z, \qquad (2.1.8)$$

$$H_y = \frac{-j\omega\varepsilon}{k_\perp^2}\partial_x E_z - \frac{jk_z}{k_\perp^2}\partial_y H_z. \qquad (2.1.9)$$

In a similar way, we use equations **(i)** and **(v)** and obtain

$$E_y = \frac{-jk_z}{k_\perp^2}\partial_y E_z + \frac{j\omega\mu}{k_\perp^2}\partial_x H_z \qquad (2.1.10)$$

$$H_x = \frac{j\omega\varepsilon}{k_\perp^2}\partial_y E_z - \frac{jk_z}{k_\perp^2}\partial_x H_z. \qquad (2.1.11)$$

Equations (2.1.8),(2.1.10) and (2.1.9),(2.1.11) can be written in a vector form

$$\boxed{\vec{E}_\perp = -\frac{jk_z}{k_\perp^2}\nabla_\perp E_z + \frac{j\omega\mu}{k_\perp^2}\vec{1}_z \times \nabla_\perp H_z} \qquad (2.1.12)$$

$$\boxed{\vec{H}_\perp = -\frac{jk_z}{k_\perp^2}\nabla_\perp H_z - \frac{j\omega\varepsilon}{k_\perp^2}\vec{1}_z \times \nabla_\perp E_z}. \qquad (2.1.13)$$

Comments:

1. The wave equations for E_z and H_z with the corresponding boundary conditions and the relations in ((2.1.12)--(2.1.13)) determine the electromagnetic field in the entire space (at any time).

2. Note that the only assumption made so far was that in the z direction the propagation is according to $\exp\left(-jk_z z\right)$. No boundary conditions have been imposed so far.

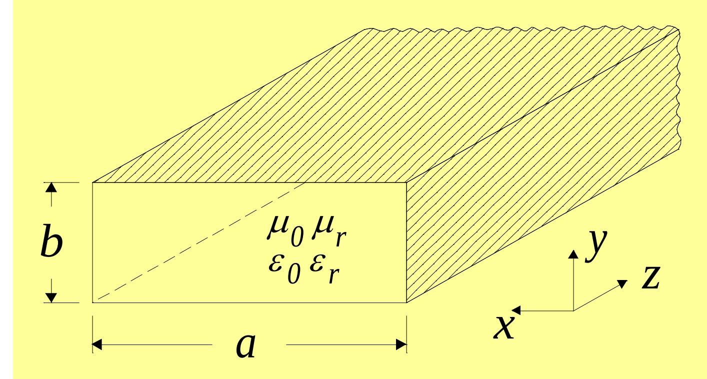
3. Therefore it is important to note within the framework of the present notation that ***TEM mode*** $(H_z = 0, E_z = 0)$ is possible, provided that $k_\perp \equiv 0$ or substituting in the wave equations

$$\nabla_\perp^2 \vec{E}_\perp = 0; \qquad \nabla_\perp^2 \vec{H}_\perp = 0. \tag{2.1.14}$$

4. By the superposition principle and the structure of ((2.1.12)--(2.1.13)), the transverse field components may be derived from the longitudinal ones.

$$\text{Complete Solution} = \underbrace{(E_z = 0)\ \text{and}(H_z \neq 0)}_{\text{Transverse Electric(TE)}} + \underbrace{(E_z \neq 0)\ \text{and}\ (H_z = 0)}_{\text{Transverse Magnetic(TM)}} .$$

2.2 Transverse Magnetic (TM) Mode [$H_z = 0$]



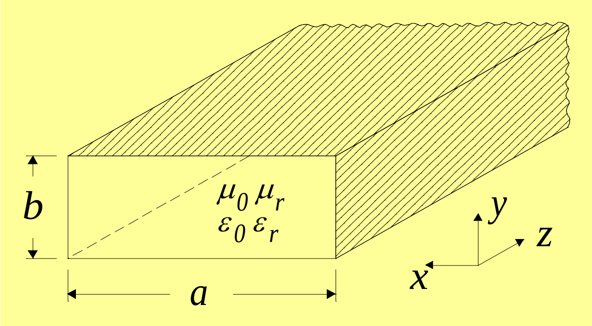
In this section our attention will be focused on a specific case where $H_z = 0$. This step is justified by the fact that equations ((2.1.12)--(2.1.13)) are linear, therefore by virtue of the superposition principle (e.g. circuit theory) and regarding H_z and E_z as sources of the transverse field, we may turn off one and solve for the other and vice versa. As indicated in the last comment of the previous section, the overall solution is, obviously the superposition of the two. The boundary conditions impose that the longitudinal electric field E_z vanishes on the metallic wall therefore

$$E_z = \sum_{n,m} A_{nm} \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) e^{-jk_{z,n,m}z}. \qquad (2.2.1)$$

This further implies that the transverse wave vector, $k_\perp$, is entirely determined by the geometry of the waveguide (substitute in (2.1.3))

$$k_\perp^2 = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 = \frac{\omega^2}{\mathrm{v}^2} - k_z^2. \qquad (2.2.2)$$

From these two equations we obtain



$$k_{z,n,m}^{\ 2} = \frac{\omega^2}{v^2} - k_\perp^2 = \frac{\omega^2}{v^2} - \left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi n}{b}\right)^2$$

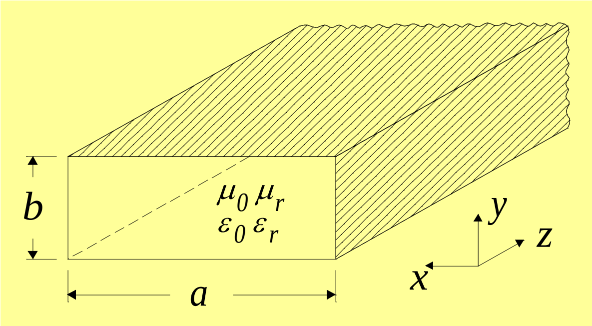
$$k_z = \pm\sqrt{\frac{\omega^2}{v^2} - \left(\frac{\pi m}{a}\right)^2 - \left(\frac{\pi n}{b}\right)^2}. \qquad (2.2.3)$$

This expression represents the ***dispersion equation*** of the electromagnetic wave in the waveguide.

Exercise 2.1: Analyze the effect of the material characteristics on the cut-off frequency.

Exercise 2.2: What is the impact of the geometry?

Exercise 2.3: Can two different modes have the same cut-off frequency?
What is the general condition for such a degeneracy to occur?



Comments:

a) Asymptotically ($\omega \gg k_\perp \mathrm{v}$) this dispersion relation behaves as if no walls were present i.e. $\omega \simeq k_z \mathrm{v}$.

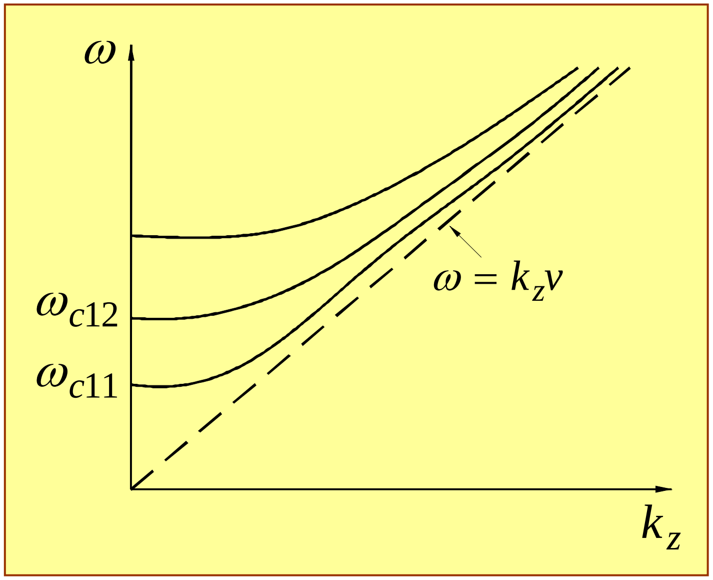
b) There is an angular frequency $\omega_{c,n,m}$ for which the wavenumber k_z **vanishes**. This is called the **cutoff frequency**.

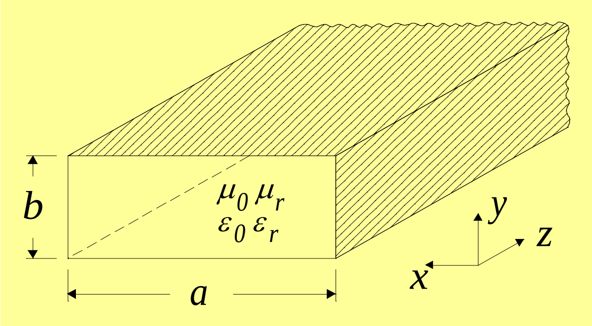
$$\omega_{c,m,n} \equiv \mathrm{v} k_\perp = \mathrm{v}\sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2} \Rightarrow (f_c)_{m,n} = \frac{1}{2}\mathrm{v}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \qquad (2.2.4)$$

where $\mathrm{v} = c/\sqrt{\varepsilon_r \mu_r}$.

c) Below this frequency the wavenumber k_z is imaginary and the wave **decays** or **grows exponentially** in space.

d) The indices n and m define the mode $TM_{m,n}$; m represents the wide transverse dimension (x) whereas n represents the narrow transverse dimension (y).





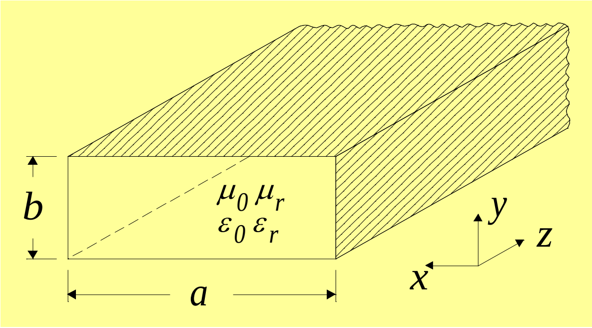
2.3 Transverse Electric (TE) Mode [$E_z = 0$]

The second possible solution according to (2.1.12)--(2.1.13) is when $E_z = 0$ and since the derivative of the longitudinal magnetic field H_z vanishes on the walls (see (2.1.12)) we conclude that

$$H_z = \sum_{m,n} A_{m,n} \cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right) e^{-jk_{z,n,m} z}. \qquad (2.3.1)$$

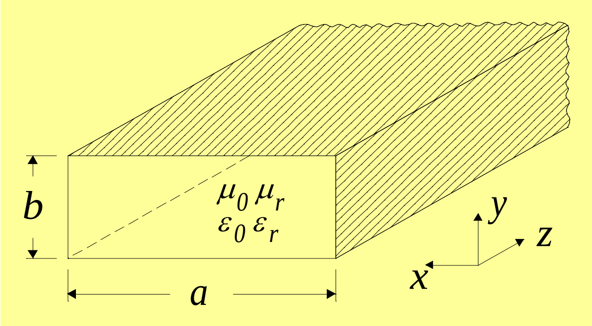
The expression for the transverse wavenumber $k_\perp$ is identical to the TM case and so is the dispersion relation. However, note that contrary to the TM mode where if n or m were zero the field component vanishes, in this case we may allow $n = 0$ or $m = 0$ without forcing a trivial solution.

For convenience, we present next a comparison table of the various field components of the two modes.



TM mode	TE mode
$E_z = \sum_{mn} A_{mn} \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) e^{-jk_{zmn}z}$	$H_z = \sum_{mn} B_{mn} \cos\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) e^{-jk_{mn}z}$
$H_x = \sum_{mn} A_{mn} \frac{j\omega\varepsilon}{k_\perp^2} \left(\frac{\pi n}{b}\right) \sin\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) \ldots$	$E_x = \sum_{mn} B_{mn} \frac{j\omega\mu}{k_\perp^2} \left(\frac{\pi n}{b}\right) \cos\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \ldots$
$H_y = \sum_{mn} A_{mn} \frac{-j\omega\varepsilon}{k_\perp^2} \left(\frac{\pi m}{a}\right) \cos\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \ldots$	$E_y = \sum_{mn} B_{mn} \frac{j\omega\mu}{k_\perp^2} \left(-\frac{\pi m}{a}\right) \sin\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) \ldots$
$H_z = 0$	$E_z = 0$
$E_x = \sum_{mn} A_{mn} \frac{-jk_z}{k_\perp^2} \left(\frac{\pi m}{a}\right) \cos\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \ldots$	$H_x = \sum_{mn} B_{mn} \frac{+jk_z}{k_\perp^2} \left(\frac{\pi m}{a}\right) \sin\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) \ldots$
$E_y = \sum_{mn} A_{mn} \frac{-jk_z}{k_\perp^2} \left(\frac{\pi n}{b}\right) \sin\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) \ldots$	$H_y = \sum_{mn} B_{mn} \frac{+jk_z}{k_\perp^2} \left(\frac{\pi n}{b}\right) \cos\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \ldots$
$E_{x,mn} = Z_{mn}^{(TM)} H_{y,mn}$; $Z_{mn}^{(TM)} = \sqrt{\frac{\mu}{\varepsilon}} \left[1 - \left(\frac{f_{c,m,n}}{f}\right)^2\right]^{1/2}$	$H_{x,mn} = -\frac{E_{y,mn}}{Z_{mn}^{(TE)}}$; $Z_{mn}^{(TE)} = \sqrt{\frac{\mu}{\varepsilon}} \left[1 - \left(\frac{f_{c,m,n}}{f}\right)^2\right]^{-1/2}$
$E_{y,mn} = -Z_{mn}^{(TM)} H_{x,mn}$	$H_{y,mn} = \frac{E_{x,mn}}{Z_{mn}^{(TE)}}$

Exercise 2.4: Check all the expressions presented in the table above.



Comments:

1. The **phase velocity** of the wave is the velocity an imaginary observer has to move in order to measure a constant phase i.e. $\omega t - k_z z = \text{const.}$. This implies,

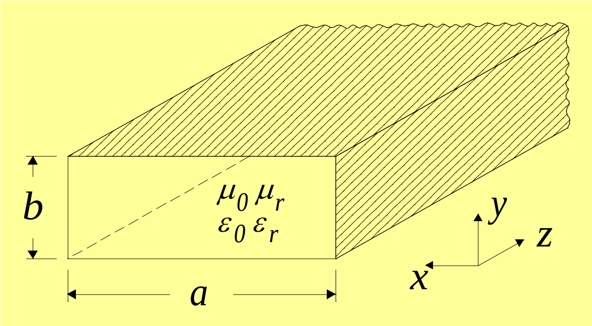
$$v_{ph} \equiv \frac{\omega}{k_z} = v\left[1 - \frac{f_c^2}{f^2}\right]^{-1/2}. \tag{2.3.2}$$

2. The phase velocity is always **faster** than v!! Specifically, in vacuum the phase velocity of a wave is larger than c. In fact close to cutoff this velocity becomes "infinite"!!
3. The **group velocity** is defined from the requirement that an observer sees a constant envelope in the case of a relatively narrow wave packet. At the continuous limit this is determined by

$$v_{gr} = \frac{\partial \omega}{\partial k_z} = v\left[1 - \frac{f_c^2}{f^2}\right]^{1/2}. \tag{2.3.3}$$

4. The group velocity is alway **smaller** than v. Specifically, in vacuum it is always smaller than c. It is the group velocity is responsible to information transfer.
5. When the waveguide is uniform

$$v_{ph} v_{gr} = v^2. \tag{2.3.4}$$



2.4 Power Considerations

2.4.1 Power Flow

Let us now consider the power associated with a specific TM mode; say

$$E_z = A_{mn} \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) e^{-jk_{z,n,m} z}.$$

At this stage, for simplicity sake, we assume that this is the **only** mode in the waveguide. Based on Poynting's theorem, the power carried by this mode is given by

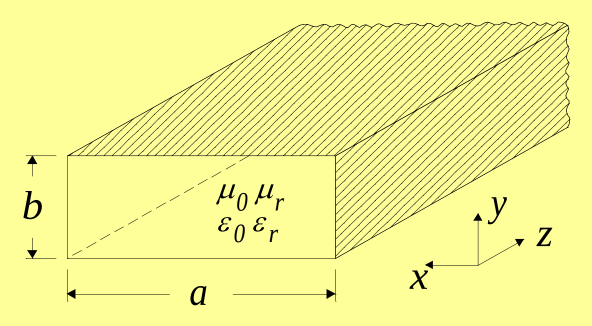
$$P_{mn} = \mathrm{Re}\left\{\int_0^a dx \int_0^b dy\, S_{z,mn}\right\}. \qquad (2.4.1)$$

Explicitly the longitudinal component of the Poynting vector is

$$S_{z,m,n} = \frac{1}{2}(\vec{E} \times \vec{H}^*) \cdot \vec{1}_z = \frac{1}{2}[E_x H_y^* - E_y H_x^*]_{m,n}$$

$$= \frac{1}{2}\left[Z^{(TM)} |H_y|^2 + Z^{(TM)} |H_x|^2\right]_{m,n}. \qquad (2.4.2)$$

Above cutoff the characteristic impedance is a real number therefore the next step is to substitute the explicit expressions for the magnetic field components and perform the spatial integration:



$$P_{mn} = \frac{1}{2} Z^{(TM)} \int_0^a dx \int_0^b dy \left[|H_y|^2 + |H_x|^2 \right]$$

$$= \frac{1}{2} Z^{(TM)} |A_{mn}|^2 \frac{\omega^2 \varepsilon^2}{k_\perp^4} \frac{a}{2} \cdot \frac{b}{2} \left[\underbrace{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}_{k_\perp^2} \right]$$

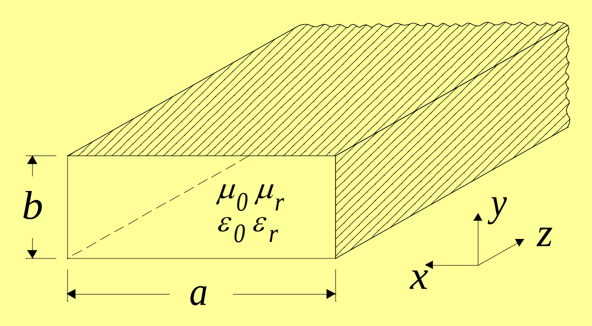
$$P_{mn} = \frac{1}{2} Z^{(TM)} |A_{mn}|^2 \frac{\omega^2 \varepsilon^2}{k_\perp^2} \frac{ab}{4}. \qquad (2.4.3)$$

The last expression represents the **average power** carried by the specific mode.

Exercise 2.5: What is the power at any particular point in time?

Exercise 2.6: Since the two sets $\sin(\pi mx / a)$ and $\sin(\pi ny / b)$ are two orthogonal sets of functions, the total average power carried by the wave in the forward direction is a superposition of the average power carried by each individual mode separately. In other words show that $P = \sum_{n,m} P_{mn}$.

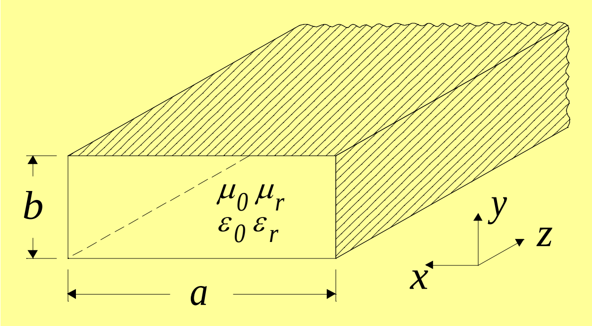
Exercise 2.7: Show that below cutoff, the power is **identically zero** although the field is **not zero**.



Exercise 2.8: Note that the average power is proportional to the average magnetic energy per unit length. Calculate this quantity. Compare it with the average electric energy per unit length.

Exercise 2.9: Calculate the energy velocity of a specific mode $v_{EM} = \langle P \rangle / W_{EM}$. Compare to the group velocity. What happens if the frequency is below cutoff?

Exercise 2.10: Repeat the last exercise for a **superposition** of modes $A_{n,m}$.



2.4.2 Ohm Loss

So far it was assumed that the walls are made of an **ideal metal** $(\sigma \to \infty)$. If this is not the case (σ) a finite amount of power is absorbed by the wall. In order to calculate this absorbed power we firstly realize that the magnetic field is "discontinuous" which is compensated by a surface current

$$\vec{J}_s = \vec{n} \times \vec{H}. \tag{2.4.4}$$

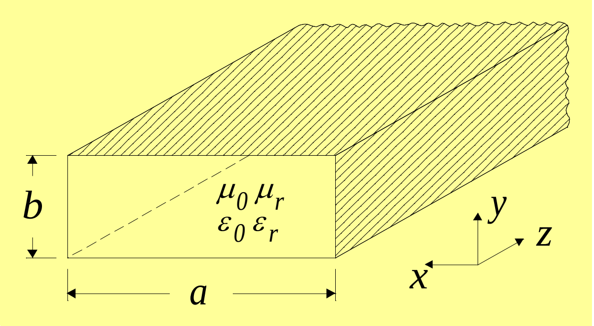
This current flows in a very thin layer which is assumed to be on the scale of the **skin-depth** $[J = J_s / \delta]$ therefore, the dissipated average power per unit length is given by

$$P_D = \frac{1}{2}\frac{1}{\sigma}\int_\delta dxdy\left|\vec{J}\right|^2, \tag{2.4.5}$$

or explicitly

$$P_D = \frac{1}{2\sigma}\oint dl\delta\left|\frac{\vec{J}_s}{\delta}\right|^2 = \frac{1}{2\delta\sigma}\oint dl\left|\vec{J}_s\right|^2 = \frac{R_s}{2}\oint dl\left|\vec{J}_s\right|^2, \tag{2.4.6}$$

where $\delta \equiv \sqrt{2/\omega\mu_o\sigma}$, $R_s \equiv \sqrt{\omega\mu_o / 2\sigma}$ and the integration is over the **circumference** of the waveguide



$$\boxed{P_D = \frac{1}{2} R_s \oint dl \left|\vec{J}_s\right|^2.} \qquad (2.4.7)$$

This is the **average electromagnetic power** per unit length which is converted into **heat** (dissipation) due to Ohm loss. Based on Poynting's theorem we may deduce that the spatial change in the electromagnetic power is given by

$$\frac{d}{dz} P = -P_D \qquad (2.4.8)$$

and since in case of a single mode both P and P_D are proportional to $|A|^2$,

$$P \propto |A|^2 \qquad \text{and} \qquad P_D \propto |A|^2 \qquad (2.4.9)$$

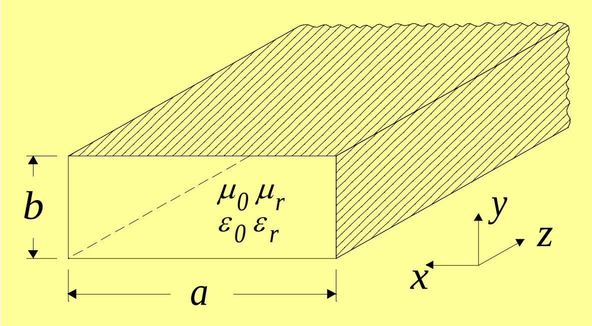
we conclude that the change in the amplitude of the mode is given by

$$\frac{d|A|^2}{dz} = -2\alpha |A|^2 \Rightarrow \quad |A(z)|^2 = |A(z=0)|^2 e^{-2\alpha z}. \qquad (2.4.10)$$

The coefficient α represents the **exponential decay** of the amplitude and based on the arguments of above is given by

$$\alpha \equiv \frac{P_D}{2P}. \qquad (2.4.11)$$

Let us denote by $k_z^{(0)}$ the wavenumber in a lossless waveguide. Subject to the



assumption of small losses ($k_z^{(0)} \gg \alpha$) we can generalize the solution in a waveguide with lossy walls by $k_z = k_z^{(0)} - j\alpha$.

Exercise 2.11: Based on the previous calculation of the power show that this parameter is given by

$$\alpha_{m,n}^{(\mathrm{TM})} = \frac{2R_s}{\eta\sqrt{1-(f_c/f)^2}}\frac{1}{b}\frac{m^2b^3+n^2a^3}{m^2b^2a+n^2a^3} \qquad (2.4.12)$$

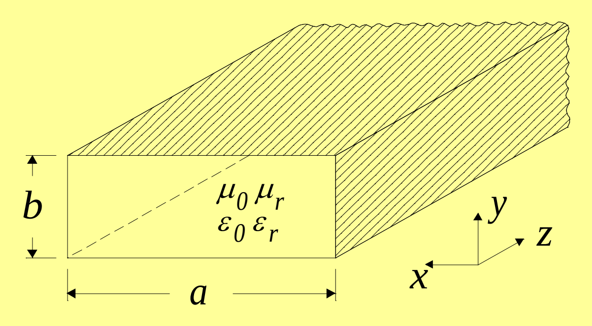
R_s is the surface resistance. Note that α is ***very large close to cutoff***. Explain the difficulty/contradiction.

2.4.3 Dielectric Loss

If the dielectric coefficient of the material is not ideal, in other words, it has an imaginary component $\varepsilon_r = \varepsilon' - j\varepsilon''$, then the wavenumber is given by

$$k_z \simeq k_z^{(0)} - j\varepsilon''\frac{1}{2}\frac{\omega}{c}\frac{\omega}{ck_z^{(0)}}, \qquad (2.4.13)$$

where we assumed that (i) the dielectric loss is small and (ii) the system operates remote from cutoff conditions [i.e. $k_z^{(0)} \gg \sqrt{\varepsilon''}\,\omega/c$].



We can now repeat the entire procedure described in the previous subsection for a TE mode. Here are the main steps and results $\left(g_0 = 1 \text{ and } g_{n\neq 0} = 2\right)$

$$H_z = B_{mn}\cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right)e^{-jk_{z,mn}z}, \quad P_{mn} = Re\left\{\int_0^a dx\int_0^b dy S_{z,mn}\right\},$$

$$S_{z,mn} = \frac{1}{2}\left(E_x H_y^* - E_y H_x^*\right)_{mn} = \frac{1}{2Z_{mn}^{(TE)}}\left[\left|E_x\right|^2 + \left|E_y\right|^2\right]_{mn}. \tag{2.4.14}$$

$$\begin{aligned} P_{mn} &= \frac{1}{2}\frac{1}{Z_{mn}^{(TE)}}\left|B_{mn}\right|^2\left[\frac{\omega^2\mu^2}{k_\perp^4}\left(\frac{\pi n}{b}\right)^2\frac{1}{2}a\frac{1}{2}b + \frac{\omega^2\mu^2}{k_\perp^4}\left(\frac{\pi m}{a}\right)^2\frac{1}{2}a\frac{1}{2}b\right] \\ &= \frac{1}{8}\frac{ab}{Z_{mn}^{(TE)}}\left|B_{mn}\right|^2\frac{\omega^2\mu^2}{k_\perp^2} \end{aligned} \tag{2.4.15}$$

$$\boxed{\alpha_{mn}^{(\mathrm{TE})} = \frac{2R_s}{\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\frac{1}{b}\left[\left(1+\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2 + \frac{b}{a}\left(\frac{g_n}{2} - \left(\frac{f_c}{f}\right)^2\right)\left(\frac{m^2ab + n^2a^2}{m^2b^2 + n^2a^2}\right)\right].} \tag{2.4.16}$$

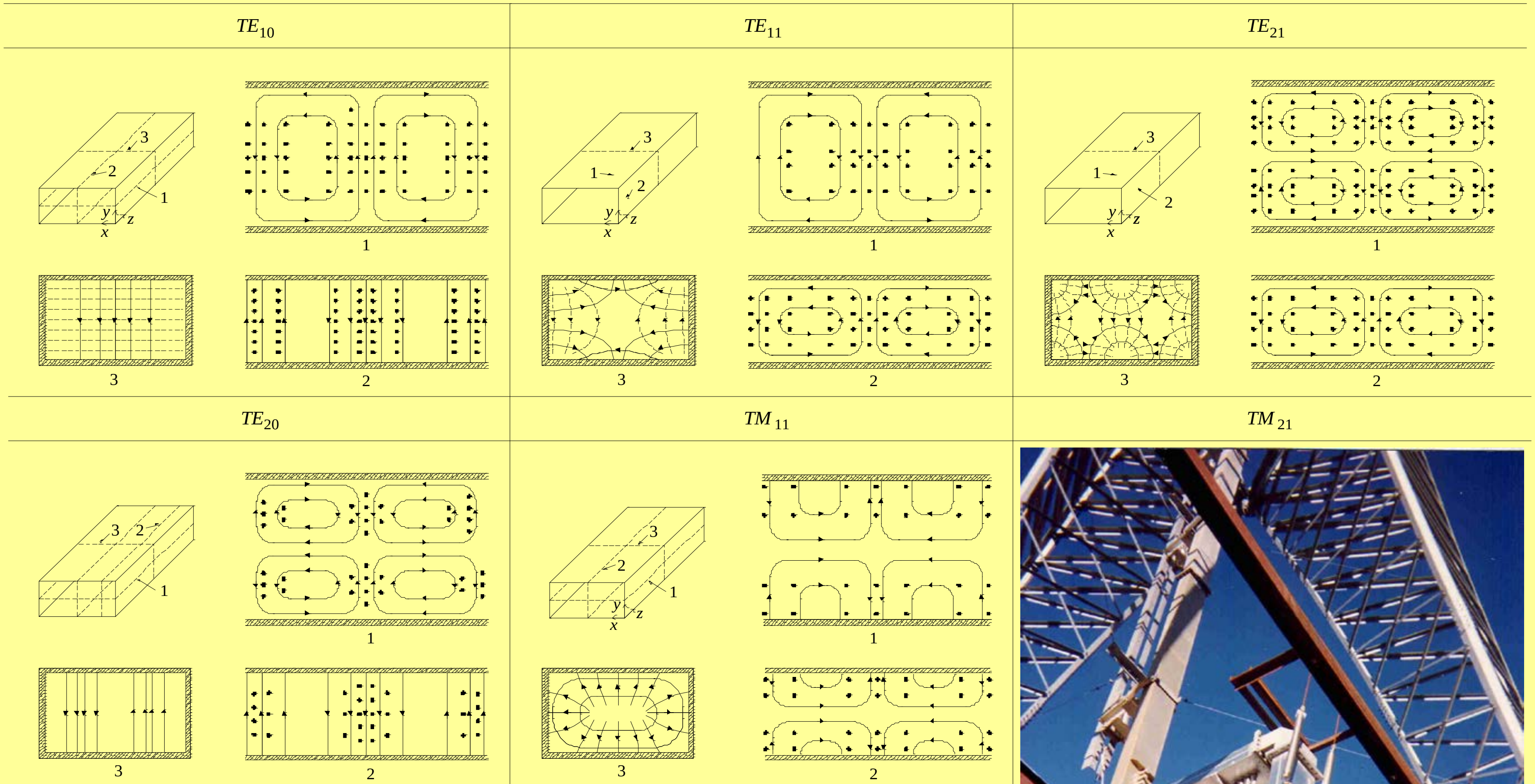
Exercise 2.12: Check equation (2.4.16). In particular check the cases $n = 0$ or $m = 0$. Repeat all the exercises from the above (TM mode) for the TE mode. Make a comparison table where relevant.

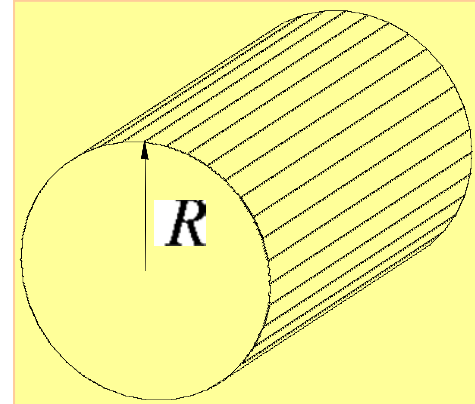
Exercise 2.13: Note that both $\alpha^{(TE)}$ and $\alpha^{(TM)}$ are large close to cutoff and increase as $\sqrt{\omega}$ for large frequencies. In between there is a **minimum loss for an optimal frequency**. Calculate it.

Exercise 2.14: Calculate the loss **very close to cutoff**.

2.5 Mode Comparison

Mode comparison for a rectangular waveguide.


TE_{10}
TE_{11}
TE_{21}
TE_{20}
TM_{11}
TM_{21}



2.6 Cylindrical Waveguide

2.6.1 Transverse Magnetic (TM) Mode $[H_z = 0]$

In this section we shall investigate the propagation of a wave in a cylindrical waveguide. The longitudinal component of the electric field satisfies

$$[\nabla_\perp^2 + k_\perp^2]E_z = 0, \tag{2.6.1}$$

where $k_\perp^2 = \dfrac{\omega^2}{\mathrm{v}^2} - k_z^2$ or explicitly

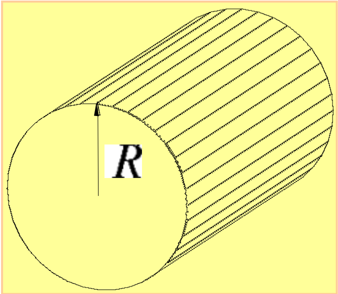
$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + k_\perp^2\right]E_z = 0. \tag{2.6.2}$$

The solution of this equation subject to the **boundary conditions** $E_z(r = R) = 0$ reads

$$E_z = \sum_{n,s} J_n\left(p_{s,n}\frac{r}{R}\right)e^{-jk_{z,s,n}z}\left[A_{n,s}\cos(n\varphi) + B_{n,s}\sin(n\varphi)\right], \tag{2.6.3}$$

where $J_n(u)$ is the n'th **order Bessel function of the first kind**. This function behaves similar to a trigonometric function $(\sin$ or $\cos)$. It has zeros, denoted by $p_{s,n}$ i.e.,

$$p_{s,n}:\ \ J_n(p_{s,n}) \equiv 0.$$



The first few **zeros** of the **Bessel function** are tabulated next.

	s=1	s=2	s=3
n=0	2.405	5.52	8.654
n=1	3.832	7.016	10.174
n=2	5.135	8.417	11.62

Substituting (2.6.3) in (2.6.2) we obtain

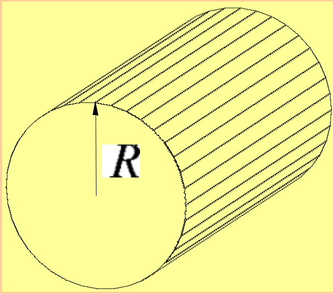
$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+\left(\frac{p_{s,n}}{R}\right)^2\right]E_z=0 \qquad (2.6.4)$$

thus

$$k_\perp^2 \equiv \frac{\omega^2}{\mathrm{v}^2}-k_z^2=\left(\frac{p_{s,n}}{R}\right)^2 \quad \Rightarrow \quad \boxed{k_{z,s,n}^2=\frac{\omega^2}{\mathrm{v}^2}-\frac{p_{s,n}^2}{R^2}}. \qquad (2.6.5)$$

Based on this expression the characteristic impedance of the TM mode is given by

$$\boxed{Z_{s,n}^{(TM)}=\eta\frac{k_{z,s,n}\mathrm{v}}{\omega}.} \qquad (2.6.6)$$



2.6.2 Transverse Electric (TE) Mode $[E_z = 0]$

In this case the wave equation reads

$$[\nabla_\perp^2 + k_\perp^2] H_z = 0 \qquad (2.6.7)$$

and its solution has the form

$$H_z = \sum_{s,n} J_n\left(p'_{s,n}\frac{r}{R}\right) e^{-jk_{zs,n}z}\left[A_{n,s}\cos(n\varphi) + B_{n,s}\sin(n\varphi)\right], \qquad (2.6.8)$$

where

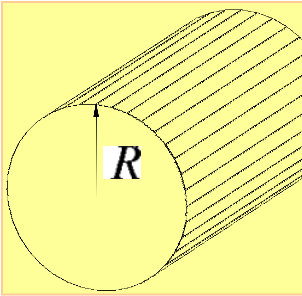
$$\left.\frac{\partial H_z}{\partial r}\right|_{r=R} = 0 \quad \Rightarrow \quad p'_{s,n} : \; J'_n(p'_{s,n}) = 0. \qquad (2.6.9)$$

The first few zeros of the **derivatives** of the Bessel function are

	s=1	s=2	s=3
n=0	3.832	7.016	10.174
n=1	1.841	5.331	8.536
n=2	3.054	6.706	9.970

thus

$$Z_{s,n}^{(TE)} = \eta\frac{\omega}{\text{v}k_{z,s,n}}, \quad k_{z,s,n} = \sqrt{\left(\frac{\omega}{\text{v}}\right)^2 - \left(\frac{p'_{s,n}}{R}\right)^2} \qquad (2.6.10)$$

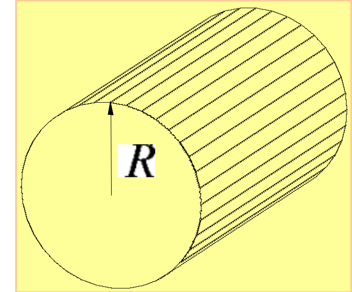


Summary

	TE modes	TM modes
$H_z =$	$J_n\left(p'_{sn}\frac{r}{R}\right)e^{-jk_{sn}z}\begin{cases}\cos n\varphi\\ \sin n\varphi\end{cases}$	0
$E_z =$	0	$J_n\left(p_{sn}\frac{r}{R}\right)e^{-jk_{sn}z}\begin{cases}\cos n\varphi\\ \sin n\varphi\end{cases}$
$H_r =$	$-\frac{jk_{sn}p'_{sn}}{Rk^2_{\perp,sn}}J'_n\left(p'_{sn}\frac{r}{R}\right)e^{-jk_{sn}z}\begin{cases}\cos n\varphi\\ \sin n\varphi\end{cases}$	$-E_\varphi / Z^{(TM)}_{sn}$
$H_\varphi =$	$-\frac{jnk_{sn}}{rk^2_{\perp,sn}}J_n\left(p'_{sn}\frac{r}{R}\right)e^{-jk_{sn}z}\begin{cases}-\sin n\varphi\\ \cos n\varphi\end{cases}$	$E_r / Z^{(TM)}_{sn}$
$E_r =$	$Z^{(TE)}_{sn}H_\varphi$	$-\frac{jk_{sn}p_{sn}}{Rk^2_{\perp,sn}}J'_n\left(p_{sn}\frac{r}{R}\right)e^{-jk_{sn}z}\begin{cases}\cos n\varphi\\ \sin n\varphi\end{cases}$
$E_\varphi =$	$-Z^{(TE)}_{sn}H_r$	$-\frac{jnk_{sn}}{rk^2_{\perp,sn}}J_n\left(p_{sn}\frac{r}{R}\right)e^{-jk_{sn}z}\begin{cases}-\sin n\varphi\\ \cos n\varphi\end{cases}$
$k_{sn} =$	$\left[\frac{\omega^2}{\mathrm{v}^2}-\left(\frac{p'_{sn}}{R}\right)^2\right]^{1/2}$	$\left[\frac{\omega^2}{\mathrm{v}^2}-\left(\frac{p_{sn}}{R}\right)^2\right]^{1/2}$
$Z^{(TE)}_{sn} =$	$\frac{\omega}{\mathrm{v}k_{sn}}\eta$	$Z^{(TM)}_{sn} = \frac{\mathrm{v}k_{sn}}{\omega}\eta$
$k_{\perp,sn} =$	p'_{sn} / R	p_{sn} / R

	TE modes	TM modes
Power=	$\frac{\eta k_0 k_{sn}\pi}{2k_{\perp,sn}^4 g_{0,n}}(p_{sn}^{'2}-n^2)J_n^2(p'_{sn})$	$\frac{\eta k_0 k_{sn}\pi}{2k_{\perp,sn}^4 g_{0,n}} p_{sn}^2[J'_n(k_{\perp,sn}R)]^2$
$\alpha =$	$\frac{R_s}{R\eta}\left(1-\frac{k_{\perp,sn}^2 \mathrm{v}^2}{\omega^2}\right)^{-1/2}\left[\frac{k_{\perp,sn}^2 \mathrm{v}^2}{\omega^2}+\frac{n^2}{p_{sn}^{'2}-n^2}\right]$	$\frac{R_s}{R\eta}\left(1-\frac{k_{\perp,sn}^2 \mathrm{v}^2}{\omega^2}\right)^{-1/2}$

$$\mathrm{v}=\frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}},\quad \eta=\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}\qquad g_{0,n}=\begin{cases}1 & n=0\\ 2 & n\neq 0\end{cases}$$



2.6.3 Power Considerations

According to Maxwell's Equations for a **single mode** we have

$$\vec{E}_\perp = -\frac{jk_z}{k_\perp^2}\nabla_\perp E_z, \qquad H_\varphi = \frac{E_r}{Z_{TM}}, \qquad H_r = -E_\varphi / Z_{TM}.$$

Consequently, the average Poynting vector is

$$S_z = \frac{1}{2}\left(\vec{E}_\perp \times \vec{H}_\perp^*\right)\cdot \vec{1}_z$$

and the average power flowing in the waveguide

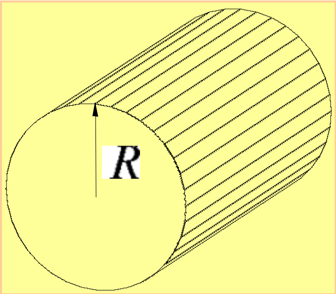
$$P = \mathrm{Re}\left\{\frac{1}{2}\int_{c.s.} da\left[\vec{E}_\perp \times \vec{H}_\perp^*\right]_z\right\} = \frac{1}{2}\mathrm{Re}\left\{\int_{c.s.} da\left[E_r H_\varphi^* - E_\varphi H_r^*\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\int_{c.s.} da\left[\frac{|E_r|^2}{Z_{TM}} + \frac{|E_\varphi|^2}{Z_{TM}}\right]\right\} = \frac{1}{2Z_{TM}}\mathrm{Re}\left\{\int_{c.s.} da\left[|E_r|^2 + |E_\varphi|^2\right]\right\}$$

$$= \frac{1}{2Z_{TM}}\mathrm{Re}\left\{\int_{c.s.} da\left(\frac{k_z}{k_\perp^2}\right)^2 |\nabla_\perp E_z|^2\right\} = \frac{1}{2Z_{TM}}\frac{k_z^2}{k_\perp^4}\mathrm{Re}\left\{\int_{c.s.} da|\nabla_\perp E_z|^2\right\}. \qquad (2.6.11)$$

In order to further simplify the last expression let us examine the wave equation:

$$\left(\nabla_\perp^2 + k_\perp^2\right)E_z = 0 \qquad (2.6.12)$$



we multiply by the complex conjugate of E_z

$$E_z^*\left(\nabla_\perp^2 + k_\perp^2\right)E_z = 0 \tag{2.6.13}$$

and integrate over the entire cross section

$$\Delta_z \int da\left[E_z^* \nabla_\perp^2 E_z + k_\perp^2 \left|E_z\right|^2\right] = 0$$

$$\Delta_z \int da\left[\nabla_\perp\left(E_z^*\nabla_\perp E_z\right) - \left|\nabla_\perp E_z\right|^2 + k_\perp^2\left|E_z\right|^2\right] = 0. \tag{2.6.14}$$

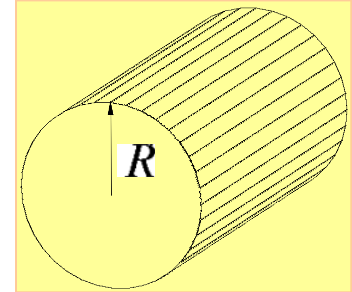
The first term in the integrand is zero since

$$\int da \nabla_\perp\left[E_z^*\nabla_\perp E_z\right] = \int d\vec{\ell}_\perp \cdot \left[E_z^*\nabla_\perp E_z\right] = 0$$

hence

$$\int da\left|\nabla_\perp E_z\right|^2 = k_\perp^2 \int da\left|E_z\right|^2. \tag{2.6.15}$$

Now back to the propagating power for a **superposition of modes** starting from the expression for a single mode we get



$$P_{s,n} = \frac{1}{2Z_{s,n}^{(TM)}}\frac{k_z^2}{k_\perp^4}\mathrm{Re}\left\{\int da\left|\nabla_\perp E_z\right|^2\right\} = \frac{1}{2Z_{s,n}^{(TM)}}\frac{k_z^2}{k_\perp^2}\int da\left|E_z\right|^2$$

$$P = \frac{1}{2\eta}\int_0^{2\pi} d\varphi\int_0^R drr\left[\sum_{s,n}\frac{k_z}{k_\perp}A_{n,s}J_n\left(p_{n,s}\frac{r}{R}\right)\cos(n\varphi)\left[\sum_{s',n'}\frac{\omega/\mathrm{v}}{k_\perp}A_{n',s'}J_{n'}\left(p_{n',s'}\frac{r}{R}\right)\cos(n'\varphi)\right]\right]$$

The integration over the angle is straight forward

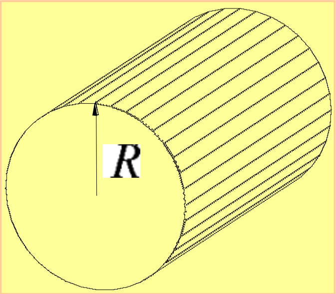
$$\int_0^{2\pi} d\varphi\cos(n\varphi)\cos(n'\varphi) = 2\pi g_n\delta_{n,n'}\ ,\ g_n = \begin{cases} 1 & n=0 \\ \frac{1}{2} & n\neq 0\end{cases}$$

and after integration over *r* we get

$$P = 2\pi\sum_{n,s,s'} g_n\frac{1}{2Z_{s,n}^{(TM)}}\frac{k_{z,s,n}^2}{k_{\perp,s,n}^2}A_{s,n}^2\frac{R^2}{2}\left[J_n'(p_{n,s})\right]^2\delta_{s,s'}$$

where we used

$$\int_0^R dr\,r\,J_n\left(p_{n,s}\frac{r}{R}\right)J_n\left(p_{n,s'}\frac{r}{R}\right) = \frac{R^2}{2}\left[J_n'(p_{n,s})\right]^2\delta_{s,s'} \qquad (2.6.16)$$

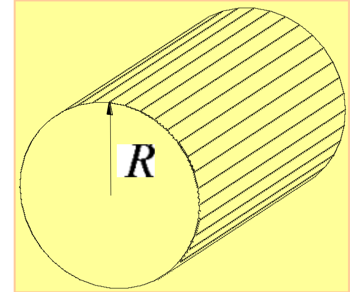


In the case of a single mode we have

$$P_{s,n} = \left(\frac{1}{2\eta}\left|RA_{s,n}\right|^2\right)\left(\frac{\omega}{\mathrm{v}}R\right)\pi g_n\left[\frac{J_n'(p_{n,s})}{p_{n,s}}\right]^2 \mathrm{Re}\left(R\sqrt{\frac{\omega^2}{\mathrm{v}^2}-\frac{p_{n,s}^2}{R^2}}\right). \tag{2.6.17}$$

Note that there is power flow only if the wave is **above cutoff** and as in the rectangular case, the the total power is the superposition of the power in each mode separately

Exercise 2.15: Calculate the average **energy per unit length** stored in the electromagnetic field.



2.6.4 Ohm Loss

Now to the general expression for the losses. Starting from the dissipated power

$$P_{D,s,n} = \frac{R_s}{2}\oint_{\text{bound}} \left|J_z^{(s)}\right|^2 d\ell = \frac{1}{2}R_s \oint \left|H_\phi\right|^2 d\ell = \frac{1}{2}R_s \oint \left|\frac{E_{r,ns}}{Z_{ns}^{(TM)}}\right|^2 d\ell$$

$$= \frac{1}{2}\frac{R_s}{\left|RZ_{ns}^{(TM)}\right|^2} k^2 R^2 \left[\frac{J_n'(p_{ns})}{p_{ns}}\right]^2 2\pi R g_n \left|A_{ns} R\right|^2$$

$$= \frac{1}{2}\frac{R_s}{\left|\eta R\right|^2}\left(\frac{\omega}{\text{v}}R\right)^2 \left[\frac{J_n'(p_{ns})}{p_{ns}}\right]^2 2\pi R g_n \left|A_{ns} R\right|^2 \qquad (2.6.18)$$

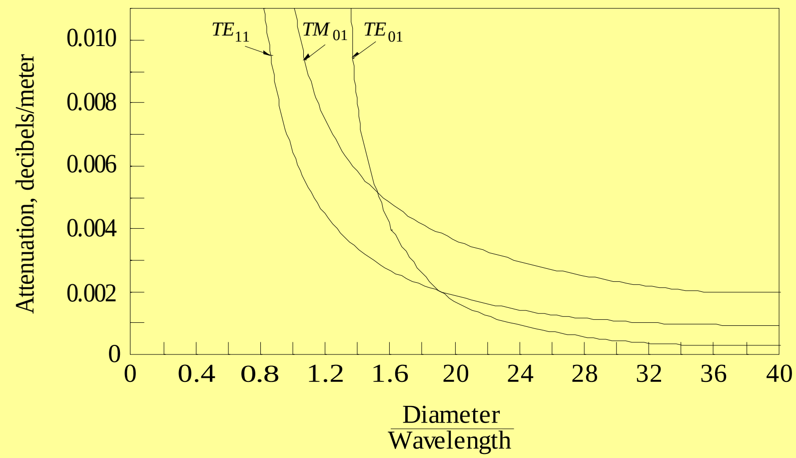
which finally entails for a **single mode**

$$\alpha_{ns}^{(TM)} = \frac{1}{R}\frac{R_s}{\eta}\left(\frac{\text{v}_{ph,sn}}{\text{v}}\right). \qquad (2.6.19)$$

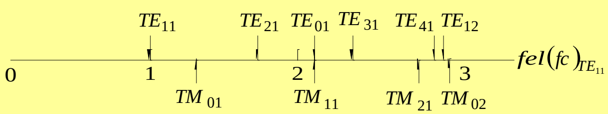
Exercise 2.16: Check Eq. (2.6.19).

Exercise 2.17: Calculate $\alpha_{ns}^{(TE)}$.

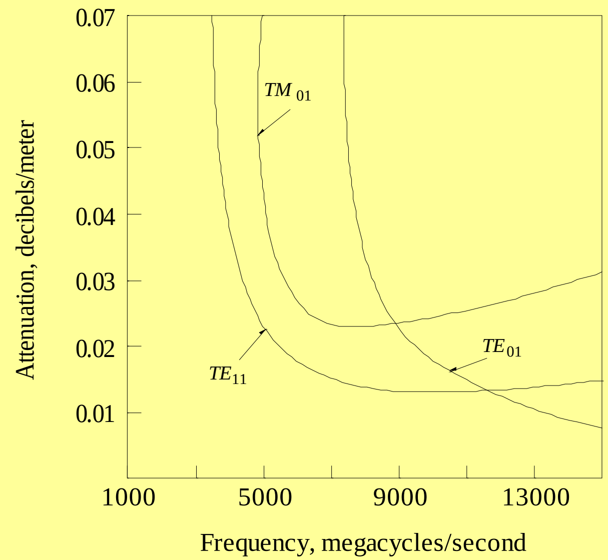
Exercise 2.18: Calculate the exponential decay due to dielectric loss.



Attenuation due to copper losses in circular waveguides at 3000 Mc/sec.



Relative cutoff frequencies of waves in a circular guide.



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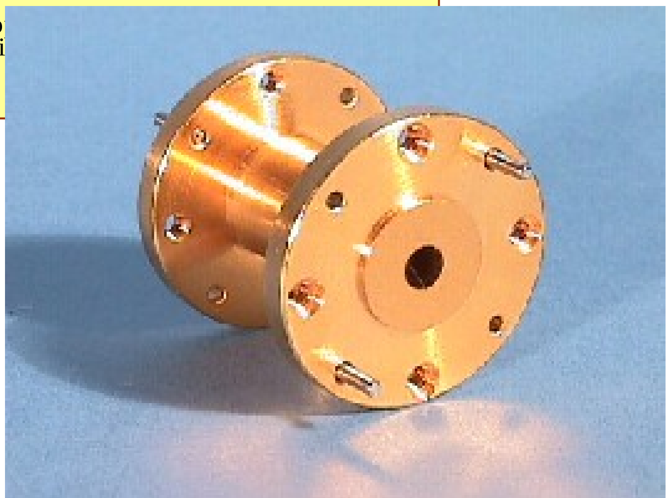
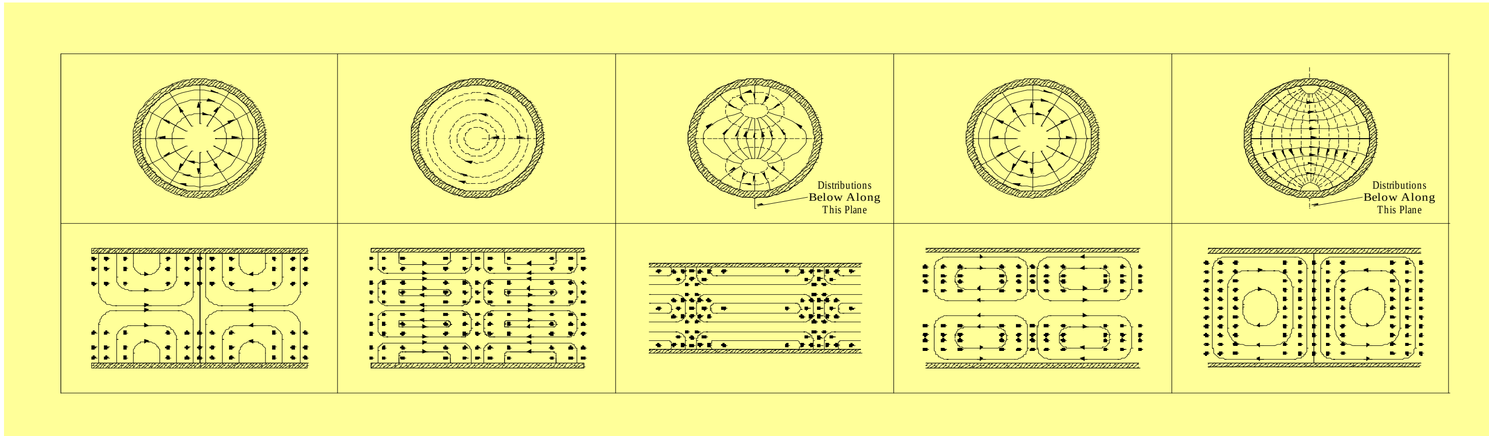
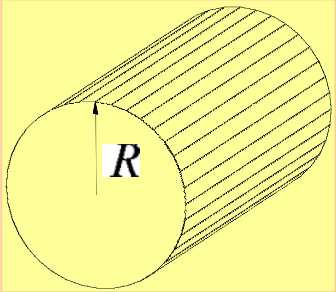
Mode comparison for cylindrical waveguide.



Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distribution in cross-sectional plane, at plane of maximum transverse fields					
Field distribution along guide					
Field components present	E_z, E_r, H_φ	E_z, E_r, H_φ	$E_z, E_r, E_\varphi, H_r, H_\varphi$	H_z, H_r, E_φ	$H_z, H_r, H_\varphi, E_r, E_\varphi$
p or p'	2.405	5.52	3.83	3.83	1.84
(k_c)	$\frac{2.405}{\alpha}$	$\frac{5.52}{\alpha}$	$\frac{3.83}{\alpha}$	$\frac{3.83}{\alpha}$	$\frac{1.84}{\alpha}$
(λ_c)	2.61α	1.14α	1.64α	1.64α	3.41α
(f_c)	$\frac{0.383}{\alpha\sqrt{\mu\varepsilon}}$	$\frac{0.877}{\alpha\sqrt{\mu\varepsilon}}$	$\frac{0.609}{\alpha\sqrt{\mu\varepsilon}}$	$\frac{0.609}{\alpha\sqrt{\mu\varepsilon}}$	$\frac{0.293}{\alpha\sqrt{\mu\varepsilon}}$
Attenuation due to imperfect conductors	$\frac{R_s}{\alpha_n}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{\alpha_n}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{\alpha_n}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{\alpha_n}\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{\alpha_n}\frac{1}{\sqrt{1-(f_c/f)^2}}$



2.7 Pulse Propagation

Let us consider an **azimuthally symmetric** TM mode described by

$$E_z(r,z,t)=\sum_{s=1}^{\infty}J_0\left(p_s\frac{r}{R}\right)\int_{-\infty}^{\infty}d\omega\exp\left[j\omega t-\Gamma_s z\right]\mathcal{E}_s(\omega) \tag{2.7.1}$$

wherein $\Gamma_s^2=p_s^2/R^2-\omega^2/c^2$ and $\mathcal{E}_s(\omega)$ is the Fourier transform of this field component at $z=0$

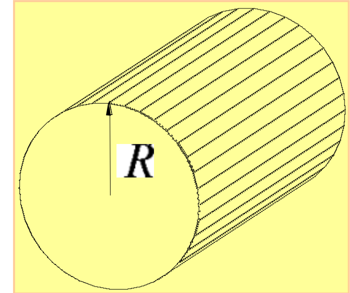
$$\mathcal{E}_s(\omega)=\frac{1}{\frac{R^2}{2}J_1^2(p_s)}\int_0^R dr'r'J_0\left(p_s\frac{r}{R}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\exp\left(-j\omega t'\right)E_z(r',z=0,t'). \tag{2.7.2}$$

Let us now calculate the **energy associated with the radiation field** as it propagates in an empty waveguide. The transverse field components are

$$E_r(r,z,t)=\sum_{s=1}^{\infty}J_1\left(p_s\frac{r}{R}\right)\frac{p_s}{R}\int_{-\infty}^{\infty}d\omega\exp\left(j\omega t-\Gamma_s z\right)\frac{\mathcal{E}_s\Gamma_s}{\Gamma_s^2+\frac{\omega^2}{c^2}}$$

$$H_\varphi(r,z,t)=\sum_{s=1}^{\infty}J_1\left(p_s\frac{r}{R}\right)\frac{p_s}{R}\int_{-\infty}^{\infty}d\omega\exp\left(j\omega t-\Gamma_s z\right)\frac{1}{\mu_0}\frac{j\omega}{c^2}\frac{\mathcal{E}_s}{\Gamma_s^2+\frac{\omega^2}{c^2}} \tag{2.7.3}$$

therefore, the z-component of the Poynting vector is



$$S_z(r,z,t) = \left\{ \sum_{s=1}^{\infty} J_1\left(p_s \frac{r}{R} \right) \int_{-\infty}^{\infty} d\omega \exp\left(j\omega t - \Gamma_s z \right) \mathcal{E}_s(\omega) \left(\Gamma_s \frac{R}{p_s} \right) \right\}$$

$$\times \left\{ \sum_{\sigma=1}^{\infty} J_1\left(p_\sigma \frac{r}{R} \right) \int_{-\infty}^{\infty} d\Omega \exp\left(j\Omega t - \Gamma_\sigma(\Omega) z \right) \mathcal{E}_\sigma(\Omega) \frac{1}{\eta_0} \left(j\Omega \frac{R}{c p_\sigma} \right) \right\}. \quad (2.7.4)$$

Using the orthogonality of Bessel functions

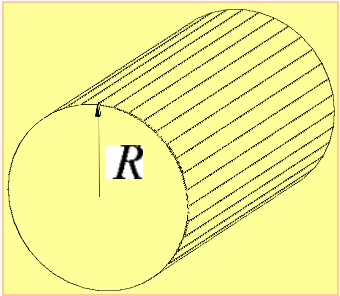
$$\int_0^R dr r J_1\left(p_s \frac{r}{R} \right) J_1\left(p_\sigma \frac{r}{R} \right) = \frac{R^2}{2} J_1^2(p_s) \delta_{s\sigma} \quad (2.7.5)$$

the power propagating is

$$P(z,t) = 2\pi \sum_{s=1}^{\infty} \frac{R^2}{2} J_1^2(p_s) \left\{ \int_{-\infty}^{\infty} d\omega \exp\left(j\omega t - \Gamma_s z \right) \mathcal{E}_s(\omega) \left(\Gamma_s \frac{R}{p_s} \right) \right\}$$

$$\times \left\{ \int_{-\infty}^{\infty} d\Omega \exp\left(j\Omega t - \Gamma_s z \right) \mathcal{E}_s(\Omega) \frac{1}{\eta_0} \left(j\Omega \frac{R}{c p_s} \right) \right\} \quad (2.7.6)$$

and the energy associated with this power



$$W_R(z) = \int_{-\infty}^{\infty} dt P(z,t)$$
$$= (2\pi)\sum_{s=1}^{\infty}\frac{R^2}{2}J_1^2(p_s)\int d\omega \mathcal{E}_s(\omega)\exp\left[-\Gamma_s(\omega)z\right]\int d\Omega \mathcal{E}_s(\Omega)\exp\left[-\Gamma_s(\Omega)z\right]$$
$$\times\left(\Gamma_s(\omega)\frac{R}{p_s}\right)\frac{1}{\eta_0}\left(j\frac{\Omega}{c}R\frac{1}{p_s}\right)\int_{-\infty}^{\infty} dt \exp\left[j(\omega+\Omega)t\right]. \quad (2.7.7)$$

With the definition of the Dirac delta function $\delta(\Omega+\omega) \equiv \frac{1}{2\pi}\int dt\, e^{j(\omega+\Omega)t}$ we have

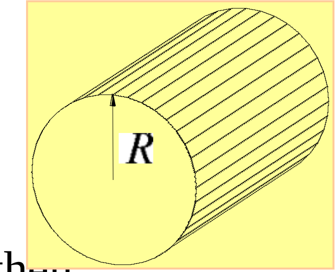
$$W_R(z) = (2\pi)^2\sum_{s=1}^{\infty}\frac{R^2}{2}J_1^2(p_s)\int_{-\infty}^{\infty} d\omega\, \mathcal{E}_s(\omega)\mathcal{E}_s(-\omega)\exp\left\{-z\left[\Gamma_s(\omega)+\Gamma_s(-\omega)\right]\right\}$$
$$\times\left(\frac{\Gamma_s(\omega)R}{p_s}\right)\frac{1}{\eta_o}\left(-j\frac{\omega}{c}R\frac{1}{p_s}\right). \quad (2.7.8)$$

For proceeding it is important to emphasize two features: since the field components are *real* functions, it is evident that the integrand ought to satisfy $\mathcal{F}_s(-\omega) = \mathcal{F}_s^*(\omega)$. Consequently

$$\mathcal{E}_s(-\omega) = \mathcal{E}_s^*(\omega) \quad (2.7.9)$$

and

$$\Gamma_s(-\omega) = \Gamma_s^*(\omega). \quad (2.7.10)$$



This last conclusion implies that if the frequency is below cut-off ($|\omega| < p_s c / R$) then

$$\Gamma_s(-\omega) = |\Gamma_s(\omega)| \qquad (2.7.11)$$

whereas if $|\omega| > cp_s / R$

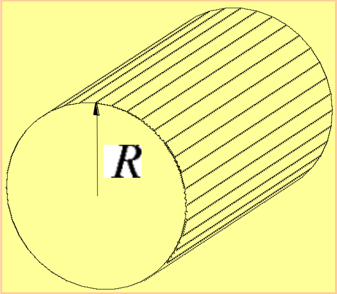
$$\Gamma_s(-\omega) = -j\,|\Gamma_s(\omega)|. \qquad (2.7.12)$$

With these observations we conclude that

$$W_R(z) = \frac{(2\pi)^2}{\eta_0}\sum_{s=1}^{\infty}\frac{R^2}{2}J_1^2(p_s)2\mathrm{Re}\left\{\int_0^{cp_s/R} d\omega\,|\mathcal{E}_s(\omega)|^2 e^{-2|\Gamma_s(\omega)|z}\right.$$
$$\left.\times\left[\frac{-j\omega R}{cp_s}\frac{|\Gamma_s(\omega)|R}{p_s}\right] + \int_{cp_s/R}^{\infty} d\omega\,|\mathcal{E}_s(\omega)|^2\left[\frac{\omega R}{cp_s}\frac{|\Gamma_s(\omega)|R}{p_s}\right]\right\}. \qquad (2.7.13)$$

Clearly the **first integrand is pure imaginary** therefore, its contribution is identically zero and as a result, in the lossless case considered here, the energy associated with the propagating signal does not change as a function of the location

$$W = \frac{(2\pi)^2}{\eta_0}\sum_{s=1}^{\infty}\frac{R^2}{2}J_1^2(p_s)\,2\int_{cp_s/R}^{\infty} d\omega\,|\mathcal{E}_s(\omega)|^2\left[\frac{\omega R}{cp_s}\frac{|\Gamma_s(\omega)|R}{p_s}\right]. \qquad (2.7.14)$$



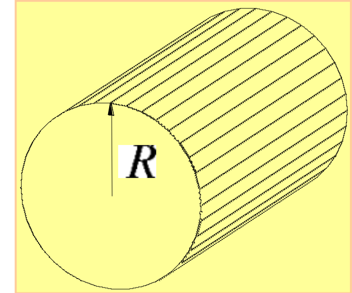
Comments:

1. Only the propagating components contribute to the radiated energy.
2. The non-propagating components are **confined** to the close vicinity of the **input** (where initial conditions were defined).
3. If all the spectrum is confined to the region $0 < \omega < p_1 c / R$, no energy will propagate.
4. Since the waveguide is lossless, the propagating energy does not change as a function of z.

Exercise 2.19: Calculate the electromagnetic energy per unit length. Compare with (2.7.14).

Let us now simplify the discussion and focus on a source which excites only the **first mode** $(s = 1)$ i.e., $E_z(r, z = 0, t) = J_0(p_1 r/R) E_0(t)$ therefore according to Eq.(2.7.2) we get

$$\mathcal{E}_s(\omega) = \frac{1}{\frac{R^2}{2} J_1^2(p_s)} \left[\int_0^R dr' r' J_0\left(p_s \frac{r}{R} \right) J_0\left(p_1 \frac{r}{R} \right) \right] \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' e^{-j\omega t'} E_0(t')$$
$$= \frac{\delta_{s,1}}{2\pi} \int_{-\infty}^{\infty} dt' e^{-j\omega t'} E_0(t') \qquad (2.7.15)$$



As an example, consider a signal starting from t=0 ramping up and oscillating and decaying

$$E_0(t) = E_0 \sin(\Omega t)\exp\left(-\frac{t}{T}\right)h(t) \Rightarrow \mathcal{E}_s(\omega) = \frac{\delta_{s,1}}{2\pi} E_0 \frac{\Omega T^2}{1+2j\omega T + T^2\left(\Omega^2-\omega^2\right)} \quad (2.7.16)$$

Substituting in Eq.(2.7.14)

$$W = \frac{1}{\eta_0 c^2} E_0^2 R^4 \left[\frac{J_1(p_1)}{p_1}\right]^2 \int_{\omega_1 = cp_1/R}^{\infty} d\omega \frac{\left(\Omega^2 T^4\right)\left(\omega\sqrt{\omega^2-\omega_1^2}\right)}{\left[1+T^2\left(\Omega^2-\omega^2\right)\right]^2 + \left(2\omega T\right)^2} \quad (2.7.17)$$

Normalizing to the cutoff frequency, $u = \omega/\omega_1$ as well as $\bar{\Omega} = \Omega/\omega_1$, $\bar{T} = T\omega_1$ the last integral simplifies to read

$$W = \varepsilon_0 E_0^2 R^3 \frac{J_1^2(p_1)}{p_1} \bar{\Omega}^2 \int_1^{\infty} du \frac{u\sqrt{u^2-1}}{\left[\bar{T}^{-2}+\bar{\Omega}^2-u^2\right]^2 + 4u^2\bar{T}^{-2}} \quad (2.7.18)$$

Its numerical analysis reveals that

$$W \sim \varepsilon_0 E_0^2 R^3 \frac{J_1^2(p_1)}{p_1} \frac{\pi}{4} \bar{\Omega}^2 \begin{cases} 1 & \bar{\Omega} < 1 \\ \bar{T} & \bar{\Omega} > 1 \end{cases} \quad (2.7.19)$$

Exercise 2.20: Compare with the dependence of the propagating energy in the frequency Ω when the signal prescribed by Eq.(2.7.16) propagates in **free space**.